Supplementary material: An Elastic Basis for Spectral Shape Correspondence

Florine Hartwig
florine.hartwig@uni-bonn.de
University of Bonn
Bonn, Germany

Josua Sassen
josua.sassen@uni-bonn.de
University of Bonn
Bonn, Germany

Omri Azencot
azencot@cs.bgu.ac.il
Ben Gurion University of the Negev
Beer Sheva, Israel

Martin Rumpf
martin.rumpf@uni-bonn.de
University of Bonn
Bonn, Germany

Mirela Ben-Chen
mirela@cs.technion.ac.il
Technion - Israel Institute of Technology
Haifa, Israel

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1 TECHNICAL COMPUTATIONS

1.1 Proofs of Lemmata

Lemma 1. Let $F \in \mathbb{R}^{m \times n}$ with $n, m > 0$ be a linear operator between two finite-dimensional Hilbert Spaces and $\| \cdot \|$ the corresponding Hilbert–Schmidt norm then

a) for all injective $\Phi_k \in \mathbb{R}^{n \times k}$, $k > 0$

$$\|F\Phi_k\|^2 = \|(\Phi_k^T F)^T\|^2 + \|\Phi_k^T (I - \Phi_k^T \Phi_k)^T\|^2$$

b) and for all injective $\Phi_k \in \mathbb{R}^{m \times k}$, $k > 0$

$$\|F\|^2 = \|F \Phi_k \|^2 + \|F (I - \Phi_k^T \Phi_k)^T\|^2.$$ 

Proof. Considering an injective $\Phi_k \in \mathbb{R}^{n \times k}$, we define $P := \Phi_k \Phi_k^T \in \mathbb{R}^{n \times n}$. We use $P^2 = P$ and $P^* = P$. This holds because $\Phi_k^T$ is an orthogonal projection with respect to the scalar product. For an explicit calculation, see Lemma 3. We have

$$\|PF\|^2 = \text{tr}((PF)^* PF) = \text{tr}(F^* PP F) = \text{tr}(F^* PF)$$

and similar

$$\|F (I - P)^T\|^2 = \text{tr}(F^* (I - P) (I - P) F) = \text{tr}(F^* (I - P) F).$$

Using the additivity of the trace, we arrive at the statement a). Statement b) follows similarly using the invariance under cyclic permutations of the trace. □

Statement b) is an orthogonal splitting of the source space of the operator $F$. For this to hold, it is important to consider the Hilbert–Schmidt norm. A weighted Frobenius norm would only reflect the correct scalar product on the target space.

Lemma 2. Let $X \in \mathbb{R}^{m \times k}$, $Y \in \mathbb{R}^{n \times k}$ be linear operators between finite-dimensional Hilbert spaces with scalar products $G_1 \in \mathbb{R}^{k \times k}$ and $G_2 \in \mathbb{R}^{n \times n}$.

a) if $G_2$ is diagonal the minimization $\min_{\Pi \in \mathbb{R}^{m \times n}} \|\Pi^T X - Y\|^2$ is row separable,

b) if $A \in \mathbb{R}^{n \times n}$ is a positive definite diagonal matrix and $G_2$ is diagonal the minimization $\min_{\Pi \in \mathbb{R}^{m \times n}} \|X^T \Pi A - Y^T\|^2$ is column separable.

To obtain Lemma 4.2 in the main text, we set $X = \Phi_{2,k} C_1^T$, $Y = \Phi_{1,k} C_1$, $G_1 = M_{1,k}$, and $G_2 = M_k$ and apply statement a). Similarly, we set $X^T = C_2^T \Phi_{2,k} M_{2}^{-1}$, $Y^T = \Phi_{1,k} C_1^T$, $G_1 = M_{1,k}$, and $G_2 = A = M_k$ and apply statement b) to obtain the corresponding statement in Section 4.3 in the main text on the dual perspective.

Proof. We first relate the Hilbert–Schmidt norm $\|F\|^2$ of a general operator $F$ between finite-dimensional Hilbert spaces with scalar products $G$ and $\tilde{G}$, respectively, to the usual Frobenius norm $\|\cdot\|_F$. This reads as

$$\|F\|^2 := \text{tr}(G^{-1} F^T \tilde{G} F) = \text{tr}(\sqrt{G^{-1}} F^T \sqrt{G} F \sqrt{G^{-1}})$$

$$= \|\sqrt{G} F \sqrt{G^{-1}}\|^2.$$ 

where $\sqrt{G}$ denotes the square root of positive-definite matrices $B$.

Applying this to the minimization in a), we can rewrite it as

$$\min_{\Pi \in \mathbb{R}^{m \times n}} \|\sqrt{G_2} \left(\Pi^T X \sqrt{G_1^{-1}} - Y \sqrt{G_1^{-1}}\right)\|^2.$$ 

As $\Pi \in \mathbb{R}^{m \times n}$, we have that each column of $\Pi \in \{0,1\}^{m \times n}$ has exactly one non-zero entry. Hence, $\Pi^T X$ is a row permutation of $X$. As $G_2$ is diagonal by assumption, the factor $\sqrt{G_2}$ is weighting the matrices row-wise and can be omitted. The minimization can then be solved...
row-wise by setting $\Pi(i, j) = 1$ if and only if

$$i = \arg\min_{r \in \{1, \ldots, m\}} \left| \sqrt[2]{G_1^{-1} \left( X^T e_r - Y^T e_j \right)} \right|_2$$

for all $j = 1, \ldots, n$, which is the same as

$$i = \arg\min_{r \in \{1, \ldots, m\}} \left| G_i^{-1} \left( X^T e_r - Y^T e_j \right) \right|_{G_i}^2.$$ 

For statement b), we rewrite the minimization as

$$\arg\min_{\Pi \in \Pi} \left| \sqrt[2]{G_1^{-1} \left( X^T \Pi A - Y^T \right)} \right|_2.$$

Now, $X^T \Pi A$ are diagonal and multiplication from the right is weighting the columns, we can solve the minimization by setting $\Pi(i, j) = 1$ if and only if

$$i = \arg\min_{r \in \{1, \ldots, m\}} \left| \sqrt[2]{G_1^{-1} \left( X^T e_r - Y^T A^{-1} e_j \right)} \right|_2$$

for all $j = 1, \ldots, n$. □

**Lemma 3 (Orthogonal Projection).** The operator $\Phi_k \Phi_k^\top \in \mathbb{R}^{n \times n}$ is self-adjoint for an injective $\Phi_k \in \mathbb{R}^{n \times k}$ with $n > k > 0$, i.e. it holds $(\Phi_k \Phi_k^\top)^* = \Phi_k \Phi_k^\top$.

**Proof.** Let us recall the definition $\Phi_k^\top = G_k^{-1} \Phi_k^\top G_k$ with $G_k = \Phi_k^\top G \Phi_k$, where $G \in \mathbb{R}^{n \times n}$ represents the scalar product of the Hilbert space. We have

$$\left( \Phi_k \Phi_k^\top \right)^* = G_k^{-1} (\Phi_k^\top)^T G_k = G_k^{-1} G_k \Phi_k G_k^{-1} \Phi_k^T G = \Phi_k \Phi_k^\top.$$ □

### 1.2 Computation of the adjoint

Computation of the adjoint (Formula (7))

$$C_{12} = M_{2,k}^{-1} \Phi_{1,k}^\top \Phi_{1,k} M_{2,k}^{-1} \Phi_{1,k}^T M_{1,k}$$

$$= \left( M_{2,k}^{-1} \Phi_{2,k} \Phi_{2,k}^T M_{2,k}^{-1} \right) M_{2,k}^{-1} \Phi_{1,k} \Phi_{1,k}^T M_{1,k}$$

$$= \left( M_{2,k}^{-1} \Phi_{2,k} \Phi_{2,k}^T M_{2,k} \right) (\Phi_{1,k} M_{1,k}^{-1} M_{1,k}) = \Phi_{1,k} \Phi_{1,k}^T M_{1,k}.$$ 

where we used $(\Phi_{1,k})^T = M_{1,k} \Phi_{1,k} M_{1,k}^{-1}$.

### 2 ADDITIONAL VISUALIZATION

#### 2.1 Additional qualitative results

In Figure 1 we give additional qualitative results for the remaining methods in Figure 5 of the main paper, see Section 5.1 for details. In Figure 2 we show a colormap representation for the experiment described in Section 5.2 of the main document. Moreover, we show the results for a shape pair with median error of our method in Figure 3. In this example the extrinsic features of the shapes vary strongly.

#### Table 1: Runtime report (in sec.)

<table>
<thead>
<tr>
<th>model (number vertices)</th>
<th>LB basis</th>
<th>Ours</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cat Lion (ca. 6k)</td>
<td>2.82/66.74</td>
<td>33.59/87.96</td>
</tr>
<tr>
<td>Homer (ca. 6.5k)</td>
<td>1.53/24.06</td>
<td>22.96/33.29</td>
</tr>
<tr>
<td>Head (ca. 15k)</td>
<td>2.31/42.29</td>
<td>38.28/131.88</td>
</tr>
</tbody>
</table>

#### 2.2 Qualitative results for different values of $k$

We show qualitative results for the iterative process initialized by a ground-truth vertex map as described in 5.4.2 in the main paper in Figure 4.

#### 2.3 Runtime analysis

We report runtime values in Table 1 for the experiments of the main document shown in Figures 5 and 6. We distinguish between the computation of the basis functions (first value) and the iterative method (second value).
Figure 1: Additional qualitative results for Figure 5 of the main paper. See Section 5.1 in main paper for details and Figure 5 of the main paper for a quantitative evaluation of these results.

Figure 2: Colormap representation of the results of Figure 8 in the main paper.
Figure 3: Correspondence of Shrec20 with median error of our method, see Section 5.4.2 for details.

Figure 4: Qualitative visualization of results of one correspondence for different values of $k$ for the experiment in Figure 10 of the main paper. We visualize the computed correspondence by showing the image of the resulting vertex map.