On the Optimality of Spectral Compression of Mesh Data

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Spectral compression of the geometry of triangle meshes achieves good results in practice, but there has been little or no theoretical support for the optimality of this compression. We show that, for certain classes of geometric mesh models, spectral decomposition using the eigenvectors of the symmetric Laplacian of the connectivity graph is equivalent to principal component analysis on that class, when equipped with a natural probability distribution. Our proof treats connected one-and two-dimensional meshes with fixed convex boundaries, and is based on an asymptotic approximation of the probability distribution in the two-dimensional case. The key component of the proof is that the Laplacian is identical, up to a constant factor, to the inverse covariance matrix of the distribution of valid mesh geometries. Hence, spectral compression is optimal, in the mean square error sense, for these classes of meshes under some natural assumptions on their distribution.

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General Terms: Theory

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1. INTRODUCTION

Triangle meshes are a popular way of representing 3D shape models. As the size and detail of the models grow, compression of the models becomes more and more important. The size of the mesh data files can be reduced by compressing either the geometry or the connectivity of the mesh, or both.

There has been much research into both geometry and connectivity coding. Most connectivity coding schemes, for example the Edgebreaker [Rossignac 1999] and the TG methods [Touma and Gotsman 1998], are based on traversing the mesh and generating code symbols representing new vertices or faces as they are traversed. The quality of the compression results from the entropy of the symbol sequence.

Typical geometry coding schemes are based on the fact that the coordinates of a mesh are not independent, and specifically, the coordinates of neighboring vertices are highly correlated, especially in smooth meshes. This correlation can be exploited by using "prediction rules"—the coordinates of a vertex are predicted from the coordinates of neighboring vertices, and only the prediction error vector is coded [Taubin and Rossignac 1998; Touma and Gotsman 1998]. The better the prediction rule is, the smaller the errors are, and the smaller the entropy of the code will be.

Spectral compression of mesh geometry [Karni and Gotsman 2000] also exploits the correlation among neighboring vertices, and implicitly applies a prediction rule that every vertex is the simple average

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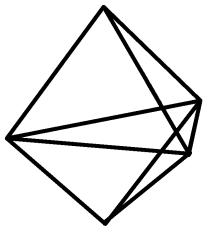


$$\begin{pmatrix} 4 & -1 & -1 & -1 & -1 \\ -1 & 4 & -1 & -1 & -1 \\ -1 & -1 & 4 & -1 & -1 \\ -1 & -1 & -1 & 3 & 0 \\ -1 & -1 & -1 & 0 & 3 \end{pmatrix}$$

Fig. 1. A simple irregular mesh and its symmetric Laplacian.

of all its immediate neighbors. Inspired by traditional signal coding, spectral decomposition has been proposed for lossy transform coding of the geometry of a mesh with irregular connectivity. Although the method yields good results in practice, there is little theoretical support for the optimality of such compression.

Motivated by the optimality of the Discrete Cosine Transform (DCT) in signal processing [Rao and Yip 1990], we wish to prove a similar result for spectral compression, namely, that it is optimal for certain classes of irregular meshes. Our proof covers connected meshes in one and two dimensions with fixed convex boundaries. The proof for the two-dimensional case is based on an asymptotic normal approximation.

1.1 Previous Work

Let G be a graph, $G = (V, E)$, where V is the vertex set, and E the edge set. A *k-dimensional mesh* M is $M = (G, R)$, $R = (X^{(1)}, X^{(2)}, \ldots, X^{(k)})$, where $X^{(i)}$ is a real vector of the i-th dimension *coordinate values* of the mesh vertices. We sometimes refer to E as the *connectivity* and to R as the *geometry* of the mesh M.

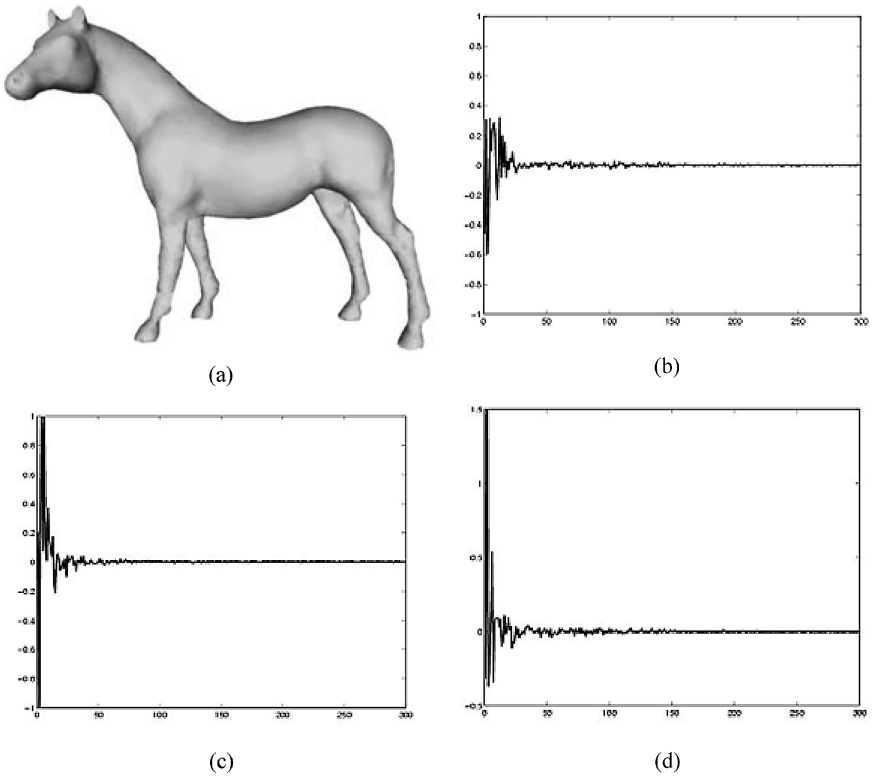
Given a mesh $M = (G, R)$, $G = (V, E)$, the *symmetric Laplacian* of M is the matrix L:

$$L_{i,j} = \begin{cases} d_i & i = j \\ 0 & (i, j) \notin E \\ -1 & (i, j) \in E \end{cases}$$

where d_i is the number of neighbors (*valence*) of the i-th vertex. A mesh with constant valences is called a *regular* mesh; otherwise, it is called an *irregular* mesh. See Figure 1 for an example of an irregular mesh, and its corresponding Laplacian.

Spectral decomposition was first introduced by Karni and Gotsman [2000] as a tool for mesh geometry compression. In their work, the eigenvectors of the Laplacian matrix are used as the basis for the decomposition of all meshes having the same connectivity. Karni and Gotsman argued that since these vectors were a generalization of the Fourier basis vectors obtained for regular (grid-type) connectivities, they should work just as well for irregular connectivities. Indeed, they showed that the coefficients of such decomposition decay rapidly to zero (see Figure 2). Hence, a small number of coefficients, combined with aggressive quantization, can be used to code the mesh geometry. The eigenvalues corresponding to the eigenvectors are analogous to the concept of frequency in harmonic analysis.

Eigenanalysis of the graph Laplacian is a standard tool in spectral graph theory [Chung 1997]. Several variants of the Laplacian have been studied. Eigenvectors of Laplacians are also used for graph drawing [Hall 1970; Koren 2003], graph partitioning [Alpert and Yao 1995], and parametrization [Gotsman et al. 2003]. For a more extensive survey of the uses of Laplacians in digital mesh processing, see Gotsman [2003]. The *inverse* of the Laplacian is closely related to *discrete Green's functions* [Chung and Yau 2000], and, as we shall see, plays a central role in this article.



(a) (b) (c) (d)

Fig. 2. Spectral decomposition. (a) The horse model and (b, c, d) the spectral coefficients of its decomposition in the X, Y, and Z dimensions.

The rest of the article is organized as follows. Section 2 defines the terminology used throughout the article. Section 3 reviews the concept of principal component analysis, a key tool in our proofs, which is the motivation for studying the eigenvectors of the covariance matrix. Sections 4 and 5 prove the optimality result for the 1D and 2D cases, respectively. To conclude, Section 6 discusses our model and explores future research directions.

2. DEFINITIONS

The *decomposition* of a vector X_{nx1}, by the orthonormal basis of $R^n : U_{nxn} = \{U_1, \ldots, U_n\}$, is:

$$X = \sum_{i=1}^{n} c_i U_i, \tag{1}$$

where $C = \{c_i\}$ are the *coefficients* of the decomposition.

The *reconstruction* of a vector X from its decomposition using the first k coefficients is:

$$X_{(U,k)} = \sum_{i=1}^{k} c_i U_i. \tag{2}$$

Note that the order of the basis vectors is important, as only a prefix of them is used in the k-th reconstruction.

Because of the orthonormality of U, the Parseval identity maintains that $\|c\| = \|X\|$. A decomposition is useful when a small number of the coefficients contain a large portion of the norm (energy) of the vector. If this is the case, those coefficients alone may be used in the reconstruction, and the Euclidean distance between the reconstructed vector and the original X will be small. This is a useful feature for compression applications.

For a specific vector X, the best basis will always be $\{X, 0, 0, \ldots, 0\}$, since then all the energy is contained in the first coefficient. Hence, there is a meaningful answer to the question "what is the optimal basis for decomposition" only if we consider an entire family of vectors (finite or infinite). In our context, the family will be all the geometries which are valid for a given mesh connectivity E. These geometries will be specified by imposing an appropriate probability distribution D, derived from E, on R^n.

If X is a random variable, we denote by $Exp(X)$ the expectation of X, by $Var(X)$ the variance of X for scalar X, and by $Cov(X)$ the covariance matrix of X for vector X.

Given a probability distribution D on R^n, we say that a vector basis U is an *optimal basis* for D, if for any other basis W, and for every $1 \leq k \leq n$:

$$Exp\left(\left(X - X_{(U,k)}\right)^2\right) \leq Exp\left(\left(X - X_{(W,k)}\right)^2\right) \tag{3}$$

where the expectation is taken over D. This is called optimality in the Mean Square Error (MSE) sense.

3. PRINCIPAL COMPONENT ANALYSIS

Approximation of random signals has been studied extensively in signal processing. A well-known optimality result, which we will rely on heavily, is related to the so-called Principal Component Analysis (PCA) procedure.

Assume a random vector $X \in R^n$, sampled from distribution D, having zero mean vector and covariance matrix C. Denote by $\{\Phi_i | i = 1, .., n\}$ the eigenvectors of C, with corresponding eigenvalues $\{\lambda_i | i = 1, \ldots, n\}$, ordered such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. The matrix Φ is called the *Principal Component Analysis* of X, sometimes known as the *Karhunen-Loeve* or *Hotelling transform* of X.

It is well known [Jolliffe 1986] that the PCA is optimal in the MSE sense, as defined in (3). Note that Φ does not depend on k—the number of vectors used for the reconstruction—so the optimal basis for reconstruction from $k + 1$ basis vectors contains the optimal basis for reconstruction from k vectors.

When the class of meshes is finite and given, containing T meshes, for example, an animation sequence of a mesh with a fixed connectivity, the natural distribution is a uniform distribution on this finite set. In this case, the PCA of that class may be computed using the numerical Singular Value Decomposition (SVD) procedure [Press et al. 1987] on a matrix of size $3n \times T$, consisting of T columns, where the i-th column contains the coordinate values of the i-th mesh in the sequence. This was proposed by Alexa and Muller [2000] for compressing animated meshes.

To compute the PCA of an infinite continuum of meshes with a given connectivity, where none of them is explicitly given, we must first make some assumptions about the distribution D of this family, and then compute the covariance matrix C of this distribution. We will do this, and then show that C is essentially identical to the inverse of the mesh Laplacian matrix, up to a constant factor. Due to this special relationship, both matrices have identical eigenvectors (in opposite order), from which our optimality theorem will follow.

4. 1D MESHES

Before we proceed, a note about the connection between the proof for one dimension, and the proof for two dimensions is in order. Basically, for both cases we prove the same theorem—that $L = \alpha C^{-1}$, for some constant α. While for the one-dimensional case the theorem can be proved relatively easily, in two dimensions we resort to asymptotic approximations.

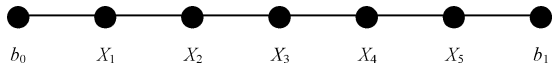


Fig. 3. The "chain" connectivity" graph for $n = 5$ in one dimension.

4.1 The Model

In one dimension, a mesh is simply an ordered set of points on the real line. There is only one "universal" (regular) connectivity—each vertex is connected to exactly two neighbors, except the two boundary vertices. For example, for $n = 5$, the connectivity graph is as shown in Figure 3.

We say that a vector $X = \{X_1, X_2, \ldots, X_n\}$ is a *valid* geometry in 1D if and only if $b_0 \leq X_1 \leq X_2 \leq \cdots \leq X_n \leq b_1$, where b_0 and b_1 are fixed boundary points. This is equivalent to saying that the "chain" connectivity does not "fold" on itself.

4.2 The Geometry Distribution

Let $U_1, U_2, \ldots, U_n$ be independent random variables, distributed uniformly on (a, b), and let $a \leq U_{(1)} \leq U_{(2)} \leq \cdots \leq U_{(n)} \leq b$ be these values after sorting them in increasing order. $U_{(1)}, U_{(2)}, \ldots, U_{(n)}$ are called *uniform order statistics* [David 1981]. We assume the valid geometries are distributed like the uniform order statistics on (b_0, b_1). Such a distribution will be uniform over all the valid geometries, because every valid geometry, that is every order statistics vector, can be generated by $n!$ permutations of the original variables, which are uniformly distributed. The mean geometry for this distribution is the grid geometry, where the distances between the vertices are equal.

4.3 The Optimal Basis

We now proceed to the main theorem in its one-dimensional version.

THEOREM 1 *The optimal basis for the decomposition of one dimensional meshes are the eigenvectors* $\{\Psi_i | i = 1..n\}$ *of the symmetric Laplacian, ordered such that the corresponding eigenvalues are* $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_n$.

As stated in Section 3, the optimal basis is the PCA, which is the matrix of the eigenvectors of the covariance matrix of the mesh geometry distribution. Let $X = \{X_1, X_2, \ldots, X_n\}$ be a random valid geometry with boundaries b_0 and b_1, and C its covariance matrix. Assume $j \leq i$, since $C_{ij} = C_{ji}$. Now by definition

$$C_{ij} = Exp(X_i X_j) - Exp(X_i)Exp(X_j).$$

The first step in finding the optimal basis is computing the covariance matrix C. In order to do that, we will need some more definitions:

Define $X_0 = b_0, X_{n+1} = b_1$, and $Y_i = X_i - X_{i-1}$ for $i = 1..n+1$. Y_i are called *uniform spacings*. It is well-known [Pyke 1965] that Y_i are interchangeable random variables, namely, that they are identically distributed, and all pairs of variables have the same joint distribution:

$$\begin{aligned} Var(Y_i) &= Var(Y_1) = v(n) \qquad \forall i \in \{1, \ldots, n+1\} \\ Cov(Y_i, Y_j) &= Cov(Y_1, Y_2) = c(n) \quad \forall i, j \in \{1, \ldots, n+1\}, i \neq j \end{aligned} \tag{4}$$

for some functions $v(n)$ and $c(n)$ which depend only on n and the boundaries b_0 and b_1 (but not on i and j).

LEMMA 1.1. *If* X *is a random valid* 1D *geometry,* C *its covariance matrix, and* $j \leq i$*, then:*

$$C_{ij} = jv(n) + (ij - j)c(n).$$

PROOF. From the definition of Y_i we have that:

$$X_i = b_0 + \sum_{k=1}^{i} Y_k. \tag{5}$$

Then, for $j \leq i$, by substituting (5) in the covariance definition, and since the Y_i are interchangeable as defined in (4), we have:

$$\begin{aligned} C_{ij} &= Exp(X_i X_j) - Exp(X_i)Exp(X_j) = \sum_{k=1}^{i}\sum_{r=1}^{j}[Exp(Y_k Y_r) - Exp(Y_k)Exp(Y_r)] \\ &= \sum_{r=1}^{j}\left[Exp\left(Y_r^2\right) - Exp(Y_r)^2\right] + \sum_{k=1}^{i}\sum_{\substack{r=1 \\ r\neq k}}^{j}[Exp(Y_k Y_r) - Exp(Y_k)Exp(Y_r)] \qquad (6) \\ &= \sum_{r=1}^{j} var(Y_1) + \sum_{k=1}^{i}\sum_{\substack{r=1 \\ r\neq k}}^{j} cov(Y_1, Y_2) = jv(n) + (ij - j)c(n). \end{aligned}$$

□

We will now relate the variance and covariance of the uniform spacings, which will allow us to simplify the expression in (6).

LEMMA 1.2. *If $v(n)$ and $c(n)$ are the variance and covariance, respectively, of uniform spacings as defined in* (4)*, then:*

$$c(n) = -\frac{1}{n}v(n).$$

PROOF. Consider the following expression:

$$Exp\left[\left(\sum_{i=1}^{n+1} Y_i\right)^2\right] - \left[Exp\left(\sum_{i=1}^{n+1} Y_i\right)\right]^2.$$

On the one hand, developing the expression, by expanding the sums and using the linearity of the expectation, yields:

$$\begin{aligned} Exp\left[\left(\sum_{i=1}^{n+1} Y_i\right)^2\right] - \left[Exp\left(\sum_{i=1}^{n+1} Y_i\right)\right]^2 &= \sum_{i=1}^{n+1}\sum_{j=1}^{n+1} Exp(Y_i Y_j) - \sum_{i=1}^{n+1}\sum_{j=1}^{n+1} Exp(Y_i)Exp(Y_j) \\ &= \sum_{i=1}^{n+1}\sum_{j=1}^{n+1}[Exp(Y_i Y_j) - Exp(Y_i)Exp(Y_j)] \\ &= \sum_{j=1}^{n+1}\left[Exp\left(Y_j^2\right) - Exp(Y_j)^2\right] \\ &\quad + \sum_{i=1}^{n+1}\sum_{\substack{j=1 \\ i\neq j}}^{n+1}[Exp(Y_i Y_j) - Exp(Y_i)Exp(Y_j)] \qquad (7) \\ &= \sum_{j=1}^{n+1} var(Y_1) + \sum_{i=1}^{n+1}\sum_{\substack{j=1 \\ i\neq j}}^{n+1} cov(Y_1, Y_2) \\ &= (n+1)v(n) + n(n+1)c(n). \end{aligned}$$

On the other hand, since the sum of the Y_i is the constant $(b_1 - b_0)$, we have:

$$Exp\left[\left(\sum_{i=1}^{n+1} Y_i\right)^2\right] - \left[Exp\left(\sum_{i=1}^{n+1} Y_i\right)\right]^2 = (b_1 - b_0)^2 - (b_1 - b_0)^2 = 0. \tag{8}$$

Comparing (7) and (8), we have:

$$(n+1)v(n) + n(n+1)c(n) = 0 \tag{9}$$

$$c(n) = -\frac{1}{n}v(n) \qquad \square$$

Note that the covariance of two spacings is always negative. Intuitively, this is because one spacing may grow only at the expense of the other spacings, since their sum is constant.

By using the two previous lemmas, we can now simplify the expression of the covariance, by substituting (9) in (6), to get:

$$C_{ij} = j(n-i+1)\frac{v(n)}{n} \text{ for every } 1 \le j \le i \le n.$$

So, the covariance matrix C is

$$C_{ij} = \begin{cases} j(n-i+1)\frac{v(n)}{n} & 1 \le j \le i \le n \\ C_{ji} & 1 \le i < j \le n \end{cases} \tag{10}$$

Now let us examine the matrix product $L \cdot C$. The valence d_i of all the (interior) vertices is 2, since every vertex has exactly two neighbors. Hence, the Laplacian is the $n \times n$ matrix:

$$L = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \ddots & \vdots \\ 0 & & \ddots & & 0 \\ \vdots & \ddots & -1 & 2 & -1 \\ 0 & \cdots & 0 & -1 & 2 \end{pmatrix}_{(n \times n)}$$

where n is the number of interior vertices.

Note that the Laplacian has entries only for the interior vertices, so the first and last rows do not sum to zero, since they belong to vertices neighboring on the boundaries. This property makes the Laplacian invertible.

By substituting (10) in the product $L \cdot C$, it is easy to see that:

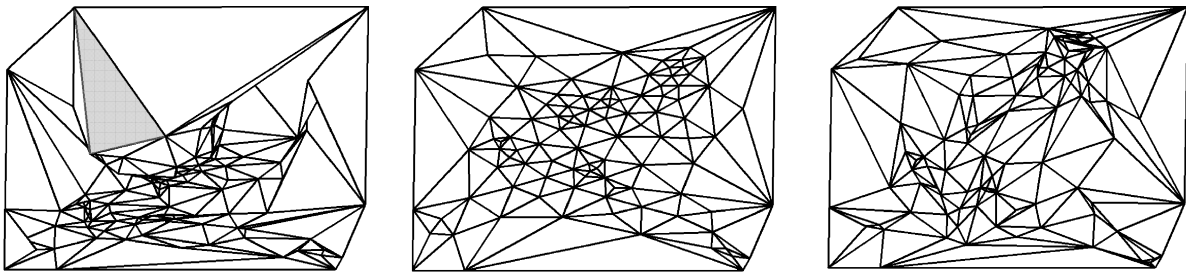
$$(L \cdot C)_{i,j} = \begin{cases} k(n) & i = j \\ 0 & i \ne j, \end{cases}$$

where $k(n)$ depends only on n, and is

$$k(n) = \frac{v(n)(n+1)}{n},$$

which implies that

$$L \cdot C = k(n)I$$



(a) (b) (c)

Fig. 4. 2D geometries with common connectivity and boundary: (a) An invalid 2D geometry. The highlighted triangle is folded over. (b, c) Valid 2D geometries.

or

$$L = k(n)C^{-1}.$$

From the optimality of PCA, we know that the eigenvectors $\{\Phi_i | i = 1..n\}$ of C, ordered such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ are the optimal basis. Since the eigenvectors of C^{-1} are the same as the eigenvectors of C, but the eigenvalues are reciprocated ($\mu_i = 1/\lambda_i$), the optimal basis for the decomposition of 1D meshes are the eigenvectors of the symmetric Laplacian, ordered so the corresponding eigenvalues are $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_n$. □

Note that the exact covariance matrix depends on $k(n)$, which depends on n and the boundaries b_0 and b_1.

5. 2D MESHES

5.1 The Model

In two dimensions, a geometry $R = (X, Y)$ is valid for the triangle mesh $M = (G, R)$, $G = (V, E)$ if the triangulation of (X, Y) using E does not contain intersecting edges, namely, the triangles do not fold over onto each other, or, in other words, all have the same orientation. Assume M has a fixed boundary, whose coordinates are

$$\begin{aligned}(X_{b_1}, X_{b_2}, \ldots, X_{b_k}) &= (x_{b_1}, x_{b_2}, \ldots, x_{b_k})\\ (Y_{b_1}, Y_{b_2}, \ldots, Y_{b_k}) &= (y_{b_1}, y_{b_2}, \ldots, y_{b_k}),\end{aligned}$$

forming a convex shape. A few examples of valid and invalid geometries for the same connectivity and convex boundaries are shown in Figure 4.

5.2 The Geometry Distribution

To prove an optimality theorem such as Theorem 1 in the two-dimensional case, we have to impose some natural probability distribution on the class of valid geometries. This is not easy, in general. For the time being, we will assume that the following three properties hold for each coordinate of the valid geometries' distribution. Later we will justify these assumptions.

(1) $X_i | X_{j \neq i} = x_j$ is normally distributed for every i, meaning that the conditional distribution of the coordinates of one vertex given the coordinates of all the others is Gaussian.

(2) $Exp(X_i | X_{j \neq i} = x_j) = \frac{1}{d_i} \sum_{j \in N(i)} x_j$, meaning that the expectation of the Gaussian distribution mentioned in 1 is the average coordinates of just the neighbors of i.

(3) $Cov(X_i, X_j \mid X_k = x_k, k \neq i, j, (i,j) \notin E) = 0$, meaning that the covariance of the coordinates of every two vertices i, j which are not neighbors, conditioned on the coordinates of all other vertices, is zero.

From now on, we will only refer to the X dimension, but all the theorems hold for the Y dimension too.

5.3 The Optimal Basis

The 2D version of our optimality theorem is:

THEOREM 2. *If the distribution of valid geometries of* 2D *meshes has the properties defined in Section* 5.2, *then the optimal basis for the decomposition of* 2D *meshes with connectivity graph* $G = (V, E)$ *are the eigenvectors* $\{\Psi_i | i = 1..n\}$ *of the symmetric Laplacian of* G, *ordered such that the corresponding eigenvalues are* $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_n$.

We first show that if the distribution of valid geometries has the first two properties described in Section 5.2, then it is multivariate normal. Note that just normal conditionals do not necessarily imply multivariate normality. However, the following Lemma characterizes the multivariate normal distribution via its conditionals:

LEMMA 2.1. (ARNOLD ET AL. 1999). *Let* $X = \{X_1, X_2, \ldots, X_n\}$ *be a random vector. If*:

(1) $X_i | X_{j \neq i} = x_j$ *is normally distributed for every* i, *and*

(2) $Exp(X_i | X_j = x_j)$ *is linear in* x_j *and not constant, for every* i,

then X *has a multivariate normal distribution.* □

It is easy to see that the conditions of the lemma are satisfied if we assume the distribution of valid geometries has the first two properties described in Section 5.2. The first condition of the lemma is identical to the first property described in Section 5.2. The second condition is implied by the second property—if the conditional expectation of a vertex is the average of its neighbors, then it is linear in x_j. In addition, since the mesh is connected, and there are no isolated vertices, the conditional expectation cannot be a constant with respect to the x_j. Thus, the second condition of Lemma 2.1 also holds. Since both the conditions of Lemma 2.1 hold, we conclude that the distribution of valid geometries (assuming Section 5.2) is multivariate normal.

Now that we have characterized the distribution of the geometries, we proceed to compute the covariance matrix, which is the key for proving optimality. The next two lemmas characterize the structure of K—the inverse of the covariance matrix. Combined, they show that K is essentially identical to the mesh Laplacian.

LEMMA 2.2. *Let* C *be the covariance matrix of the* X *component of a random valid geometry* $R = (X, Y)$ *of a mesh* $M = (G, R), G = (V, E)$. *Let* $K = C^{-1}$. *Then for every* (i, j), such that $i \neq j$, *and* $(i, j) \notin E, K_{i,j} = 0$.

PROOF. We need the following lemma which describes a few known properties of the inverse covariance matrix K of a multinormal distribution:

LEMMA 2.3 (LAURITZEN 1996). *Let* X *be a multivariate normal random variable* $X \sim N(\mu, \Sigma)$, *and let* $K = \Sigma^{-1}$. *Then*:

(1) $K_{ij} = -Cov(X_i, X_j | X_{k \neq i,j} = x_k)(K_{ii}K_{jj} - K_{ij}^2)$.

(2) $Exp(X_i | X_{j \neq i} = x_j) = Exp(X_i) + \sum_{j \neq i} \beta_{ij}(x_j - Exp(X_j))$, $\beta_{ij} = -\frac{K_{ij}}{K_{ii}}$.

Part 1 of Lemma 2.3, and the vanishing conditional covariance described in the third property of Section 5.2 imply that if i and j are not neighbors, then $K_{ij} = 0$. □

For the entries of K corresponding to the vertices and edges of the connectivity graph, we need the following Lemma.

LEMMA 2.4. *Let C be the covariance matrix of the X component of a random valid geometry $R = (X, Y)$ of a mesh $M = (G, R)$, $G = (V, E)$.* Let $K = C^{-1}$. *Then, there exists a constant α, such that*:

(1) *For every $(i, j) \in E$, $K_{ij} = -\alpha$*
(2) *For every $i \in V$, $K_{ii} = \alpha d_i$*

PROOF. From part 2 of Lemma 2.3, we know that

$$Exp(X_i | X_{j \neq i} = x_j) = Exp(X_i) + \sum_{j \neq i} \beta_{ij} (x_j - Exp(X_j)).$$

On the other hand, we assumed in the second property of Section 5.2 that

$$Exp(X_i | X_{j \neq i} = x_j) = \frac{1}{d_i} \sum_{j \in N(i)} x_j.$$

Since the linear coefficients of x_j must be equal in both expressions, we have

$$\beta_{ij} = -\frac{K_{ij}}{K_{ii}} = \frac{1}{d_i}. \tag{11}$$

C is a covariance matrix, so both C and K are symmetric, hence

$$\frac{K_{ii}}{d_i} = -K_{ij} = -K_{ji} = \frac{K_{jj}}{d_j} \tag{12}$$

for every $(i, j) \in E$.

Consider the diagonal of K. It is easy to see that if the mesh is connected, all the values K_{ii}/d_i must equal a constant α that does not depend on i: Define $K_{11}/d_1 = \alpha$. From (12) we have that for all the neighbors $j \in N(1)$, $K_{jj}/d_j = K_{11}/d_1 = \alpha$. The same can be done inductively for the neighbors of j and so on. Finally, since the mesh is connected, every vertex i has a path to the first vertex, so it must hold that:

$$\frac{K_{ii}}{d_i} = \alpha \tag{13}$$

for every $i \in V$. Substituting (13) in (12) implies

$$K_{ij} = -\alpha \tag{14}$$

for every $(i, j) \in E$ □

Combining Lemmas 2.2 and 2.4, we conclude:

Let C be the covariance matrix of the X component of a random valid geometry $R = (X, Y)$ of a mesh $M = (G, R)$, $G = (V, E)$. Let $K = C^{-1}$. Then K is:

$$K_{ij} = \begin{cases} -\alpha & (i, j) \in E \\ \alpha d_i & i = j \\ 0 & otherwise \end{cases}$$

Returning to the definition of the symmetric Laplacian in Section 1.1, we see that:

$$C^{-1} = K = \alpha L,$$

where α is a constant that depends on n alone.

As in the one-dimensional case, this concludes the proof of Theorem 2. □

Note that the exact values of the covariance matrix depend on α, which depends on the fixed boundary values.

Our theorem makes some powerful assumptions on the distribution of valid 2D geometries. We will now show why it is reasonable to make such assumptions, by describing a natural model for generating valid 2D geometries, which turns out to have a distribution with the required properties.

Following Tutte [1963], Floater [1997] proved that a 2D geometry with a convex boundary is valid if and only if each vertex is a convex combination of its neighbors. This implies that a geometry (X, Y) with a convex boundary (B_x, B_y) is valid, if and only if there exists a matrix W such that

$$\begin{aligned} X &= WX + B_x, \\ Y &= WY + B_Y \end{aligned} \tag{15}$$

where B_x and B_y are

$$B_{xi} = \begin{cases} x_i & i \in \{b_1, \ldots, b_k\} \\ 0 & otherwise \end{cases} \qquad B_{yi} = \begin{cases} y_i & i \in \{b_1, \ldots, b_k\} \\ 0 & otherwise \end{cases}$$

and W is

$$W_{ij} = \begin{cases} w_{ij} & (i, j) \in E, i \notin \{b_1, \ldots, b_k\} \\ 0 & otherwise \end{cases}.$$

The weights w_{ij} are positive and normalized:

$$\sum_{i=1}^{n} W_{ij} = 1 \quad i \notin \{b_1, \ldots, b_k\}.$$

We will call W a *barycentric coordinates matrix.*

This characterization of valid 2D meshes yields a convenient way to define a probability distribution on them. Instead of specifying the distribution of the valid geometries X (which seems to be quite difficult), we specify the distribution of the barycentric coordinates matrices. We assume that the barycentric coordinates matrices are distributed as follows:

For each interior vertex i, with valence d_i, let

$$w_{ij} = D^i_j = U^i_{(j+1)} - U^i_{(j)}, \tag{16}$$

where $U^i_{(j)}$ are $d_i - 1$ order statistics over (0, 1), with $U^i_{(0)} = 0$, and $U^i_{d(i)} = 1$. D^i_j are known as *uniform spacings* [Pyke 1965]. This guarantees that the nonzero W_{ij} are indeed positive and all the internal vertices' rows sum to one.

Note that such a distribution is not guaranteed to generate a uniform distribution of valid geometries. Since barycentric coordinates are not unique, that is, more than one set of barycentric coordinates can generate the same geometry, the use of barycentric coordinates may introduce a bias which will prefer certain valid geometries over others.

We now address the following question: If the barycentric coordinates matrices W are distributed as in (16), how are the geometries distributed? Given the matrix W and the boundary B, the geometry X

can be expressed as

$$X = (I - W)^{-1}B,$$

where I is the identity matrix.

LEMMA 2.5. *Let X be the x coordinate of a random valid geometry, whose barycentric coordinates are distributed as in* (16). *Then the distribution of X has the following properties*:

(1) *The limit distribution of $X_i|X_{j\neq i} = x_j$ as $d_i \to \infty$ is normal for every i,*

(2) $Exp(X_i|X_{j\neq i} = x_j) = \frac{1}{d_i}\sum_{j\in N(i)} x_j$,

(3) $Cov(X_i, X_j|X_k = x_k, k \neq i, j) = 0$ *for every two vertices i, j which are not neighbors.*

PROOF. From the definition of X and W in (15) and (16), respectively, it is easy to see that the conditioned variables $(X_i|X_j = x_j)$ are

$$(X_i|X_j = x_j) = \sum_{j\in N(i)} D_j^i x_j, \tag{17}$$

where D_j^i are uniform spacings, $N(i)$ is the set of neighbors of the i-th vertex, and the x_j are constants. The central limit theorem for functions of uniform spacings [Pyke 1965] implies that for vertices with large valence:

$$(X_i|X_j = x_j) \sim Normal\left(\mu_i, \sigma_i^2\right) \tag{18}$$

where *Normal* is the Gaussian distribution. This proves the first part of the Lemma.
Since uniform spacings are interchangeable random variables which sum to unity:

$$\sum_{j=1}^{d_i} D_j^i = 1 \qquad \forall i$$

it follows that

$$Exp\left(D_j^i\right) = Exp\left(D_1^i\right) = \frac{1}{d_i} \qquad \forall j. \tag{19}$$

Substituting the expectation of the spacings (19), in (17), we obtain

$$\mu_i = Exp(X_i|X_j = x_j) = Exp\left(\sum_{j\in N(i)} D_j^i x_j\right) = \sum_{j\in N(i)} Exp\left(D_j^i\right)x_j = \frac{1}{d_i}\sum_{j\in N(i)} x_j. \tag{20}$$

Note that the x_j are constants, since they are conditioned upon, so $Exp(D_j^i x_j) = Exp(D_j^i)x_j$.
This proves the second part of the Lemma.
Let i, j be two nonadjacent vertices. Consider the covariance of X_i and X_j conditioned on the rest of the vertices:

$$\begin{aligned} Cov\left(X_i, X_j|X_k = x_k, k \neq i, j\right) &= Exp(X_iX_j|X_k = x_k) - Exp(X_i|X_k = x_k)Exp(X_j|X_k = x_k) \\ &= Exp\left(\sum_{r\in N(i)} D_r^i x_r \sum_{m\in N(j)} D_m^j x_m\right) - Exp\left(\sum_{r\in N(i)} D_r^i x_r\right)Exp\left(\sum_{m\in N(j)} D_m^j x_m\right) \\ &= \sum_{r\in N(i)}\sum_{m\in N(j)} Exp\left(D_r^i x_r D_m^j x_m\right) - \sum_{r\in N(i)}\sum_{m\in N(j)} Exp\left(D_r^i x_r\right)Exp\left(D_m^j x_m\right). \end{aligned}$$

Since i and j are not neighbors, D_r^i and D_m^j are disjoint sets of independent uniform spacings, which implies that:

$$Exp\left(D_r^i x_r D_m^j x_m\right) = Exp\left(D_r^i x_r\right) Exp\left(D_m^j x_m\right) \tag{21}$$

and thus,

$$Cov(X_i, X_j | X_k = x_k, k \neq i, j) = 0 \tag{22}$$

for every nonadjacent i and j.
This proves the third part of the Lemma and concludes its proof. □

We now have a model for generating valid 2D meshes, which yields a distribution that has the properties required in Section 5.2, with just one problem—the first property of Section 5.2 requires a normal conditional distribution, and all we have is a normal limit distribution as $d \to \infty$. Central limit theorems with asymptotic parameter $d \to \infty$ give very good approximations already for modest values of d. Here the asymptotic parameter is the valence, which is 6, on the average. The following experimental results show that it seems to be large enough for the normal approximation to be reasonable.

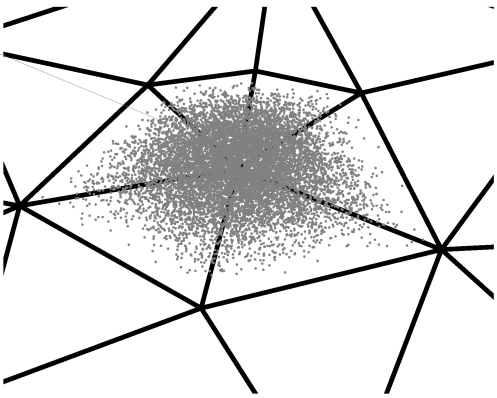
The Gaussian distribution of the coordinates can be seen in Figure 5, where some values of the coordinates of a vertex are sampled from their distribution and visualized as point clouds, as normal QQ-Plots [Rice 1988], and as a histogram.

Figure 6 visualizes another experimental test of the conditional distribution of the vertices. Here, for each vertex in the mesh, the conditional distribution for the vertex was calculated by fixing all its neighbors, and randomizing the vertex' values. Then a Bera-Jarque test for normality [Bera and Jarque 1981] was performed for each vertex. The symbol used to draw each vertex indicates what p value the vertex received in the normality test.

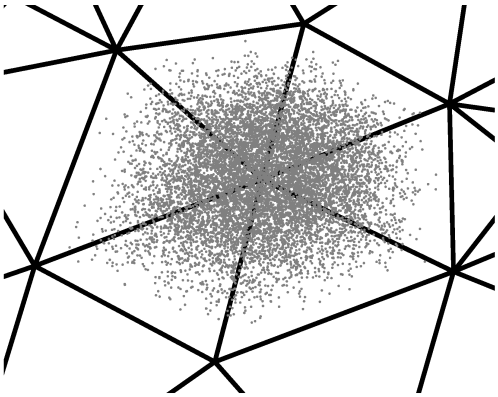
Since our result is an asymptotic result, an immediate question arises as to how good this approximation is in practice. Figures 7 and 8 show some experimental results, which give evidence to support the quality of the approximation. Both figures compare the results obtained with the symmetric Laplacian to the results obtained by an empirical covariance matrix. The empirical covariance matrix is computed by generating a random geometries matrix $X = \{X_1, \ldots, X_m\}$, where each X_i is a geometry sampled from the distribution described above. Then we compute $COV(X) = Exp(XX^T) - Exp(X)Exp(X^T)$, where $Exp(X)$ is the mean vector of X. Figure 7 shows a color-coded map of the difference matrix $|L^{-1} - \alpha C|/|\alpha C|$, where the division is performed element-by-element. The darker the color, the closer this difference is to zero. It is evident that the two matrices are practically identical. Figure 8 shows the normalized energy contained in the first k coefficients for the spectral decomposition of a geometry vector, and the decomposition using the (eigenvectors of the) empirical covariance matrix. The higher this energy is, the fewer coefficients we need to use to express more features of the geometry. It can be seen that the spectral and the PCA bases give very similar results.

A fundamental question is whether the distribution we have described above also applies to 3D meshes. It will be difficult to answer this question. Instead, we investigated the similarity between the population of 2D meshes governed by our distribution, and 3D meshes found in practice. For this, we appeal to the following simple observation: the effectiveness of the spectral compression method is

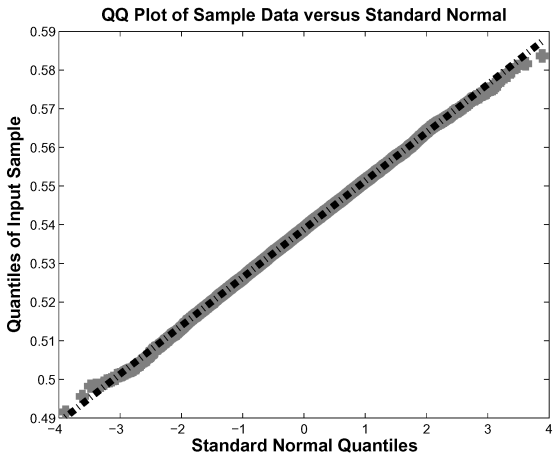
Fig. 5. The distribution of a single vertex coordinate in a random 2D mesh, using the distribution of (16). Only one vertex is sampled, while its neighbors and the rest of the mesh are fixed. (a)-(c) represent the distribution of one vertex, and (d)-(f) represent the distribution of another vertex. (a), (d): Each point in the gray cloud represents the coordinates of this vertex in a random valid geometry. The mesh connectivities are drawn in black lines using the mean valid geometry for the given connectivities (zoomed in from the full mesh). (b), (e): The distribution of the x coordinate is visualized by normal QQ-plots. In such a plot, a normal probability distribution will appear linear. (c), (f): The distribution of the x coordinate is visualized by histograms.



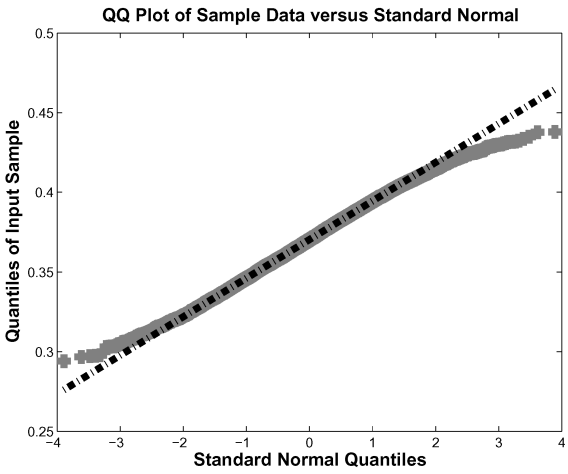
(a)



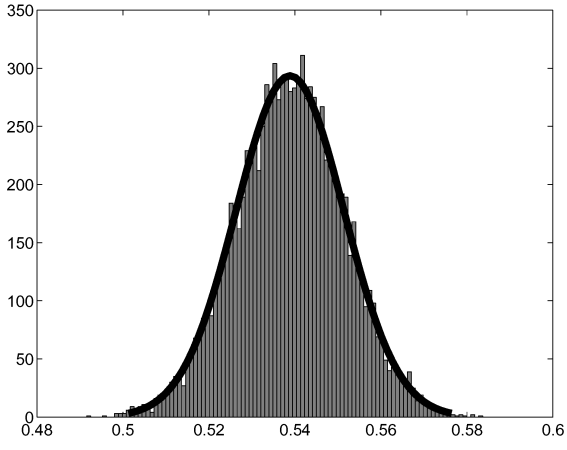
(d)


QQ Plot of Sample Data versus Standard Normal
Quantiles of Input Sample
Standard Normal Quantiles

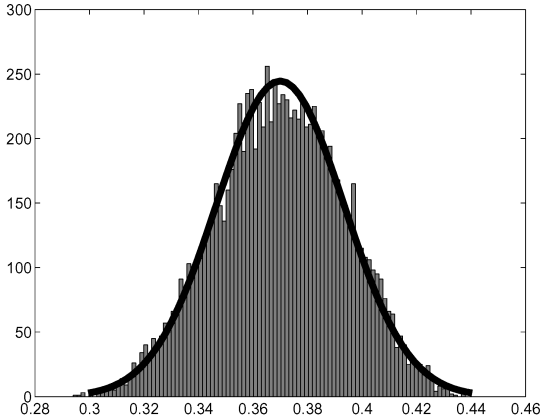
(b)


QQ Plot of Sample Data versus Standard Normal
Quantiles of Input Sample
Standard Normal Quantiles

(e)



(c)



(f)

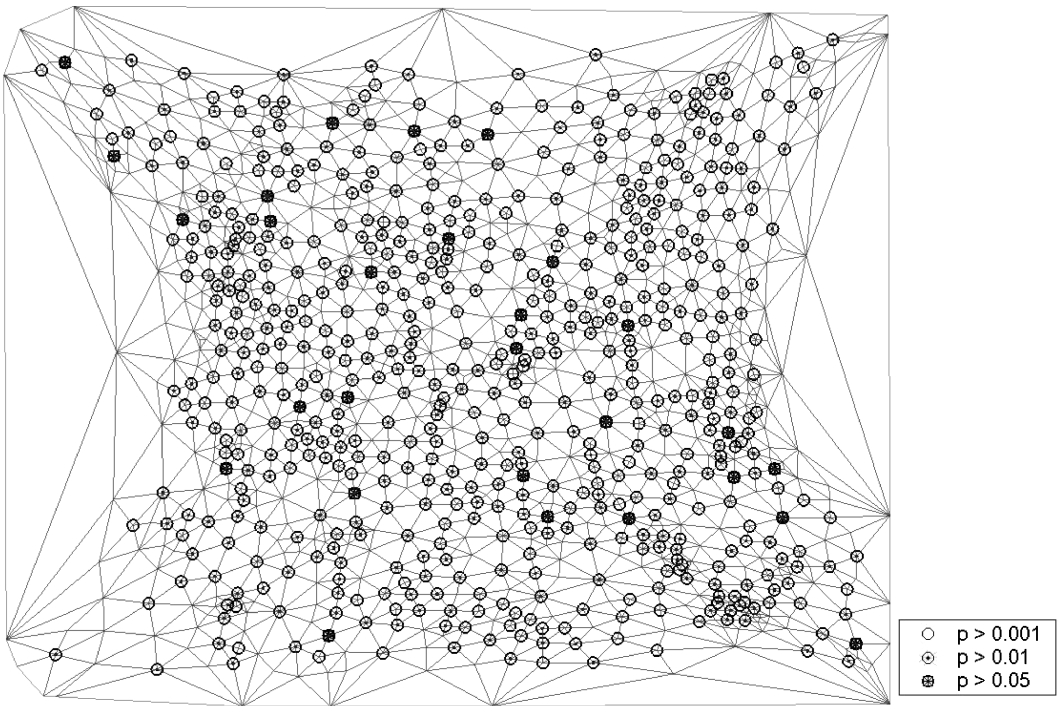


Fig. 6. For each vertex in the mesh, the conditional distribution of that vertex was measured by fixing all its neighbors, and randomizing the vertex' values. The Bera-Jarque test for normality [Bera and Jarque 1981] was performed. The vertices are marked according to the resulting p-values. Unmarked vertices have $p < 0.001$.

based on the fact that, in practice, a vertex will be close to the average of its neighbours. For example, a mesh geometry in which each vertex is located exactly at the average of its neighbours, contains no information beyond the locations of the boundary vertices. Indeed, this is also the nature of the mean of our 2D distribution. Hence, the distribution of the quantity "the normalized distance between a vertex and the average of its neighbours" across a mesh is a key measure of the efficiency of spectral compression on this mesh. We have computed and compared the distribution of this quantity for a number of well-known 3D meshes, and also for 2D meshes sampled from our distribution. Figure 9 shows that these distributions are all very similar. While this does not prove anything about the more subtle properties of the distributions, these results lend some support to our hope that our distribution also reflects well the important properties of the 3D mesh distribution.

It is worth mentioning that defining multivariate normal distributions by specifying the coefficients of the conditional means is a well-known model [Besag and Kooperberg 1995], usually referred to as *conditional autoregressive*, or *auto-normal formulation*. Such models frequently arise in Bayesian networks, where they describe interactions between random variables that have neighborhood relationships.

6. CONCLUSION, DISCUSSION, AND FUTURE WORK

We have shown that under a few natural assumptions on the distributions of geometries, given 1D and 2D connectivities, the covariance matrix of the mesh geometries is the inverse of the symmetric Laplacian matrix of that connectivity, and hence has the same eigenvectors (in reversed order). This implies that spectral decomposition of meshes, which uses the eigenvectors of the symmetric Laplacian as the basis, is optimal.

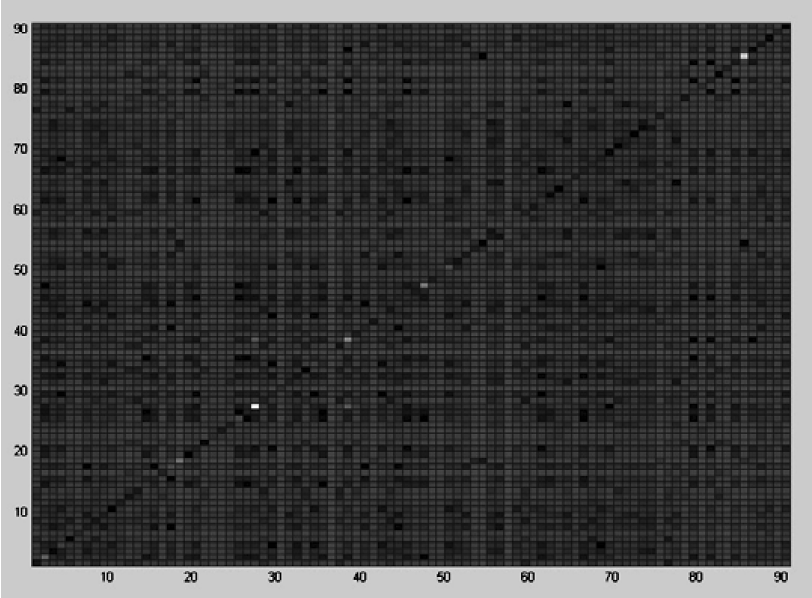


Fig. 7. The difference matrix $|L^{-1} - \alpha C|/|\alpha C|$, color-coded. The darker the color, the closer to zero the value in the difference matrix. The maximal (white) entry is approximately 3.4.

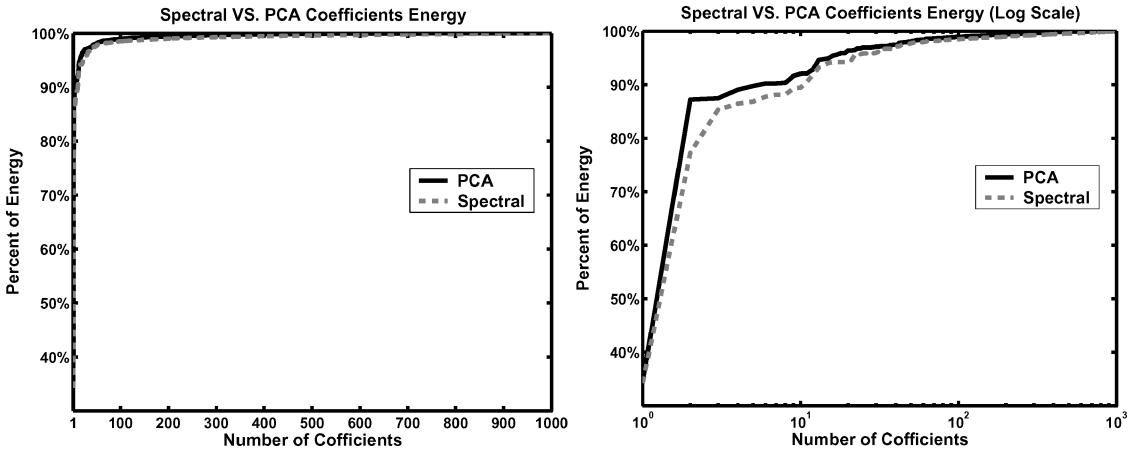
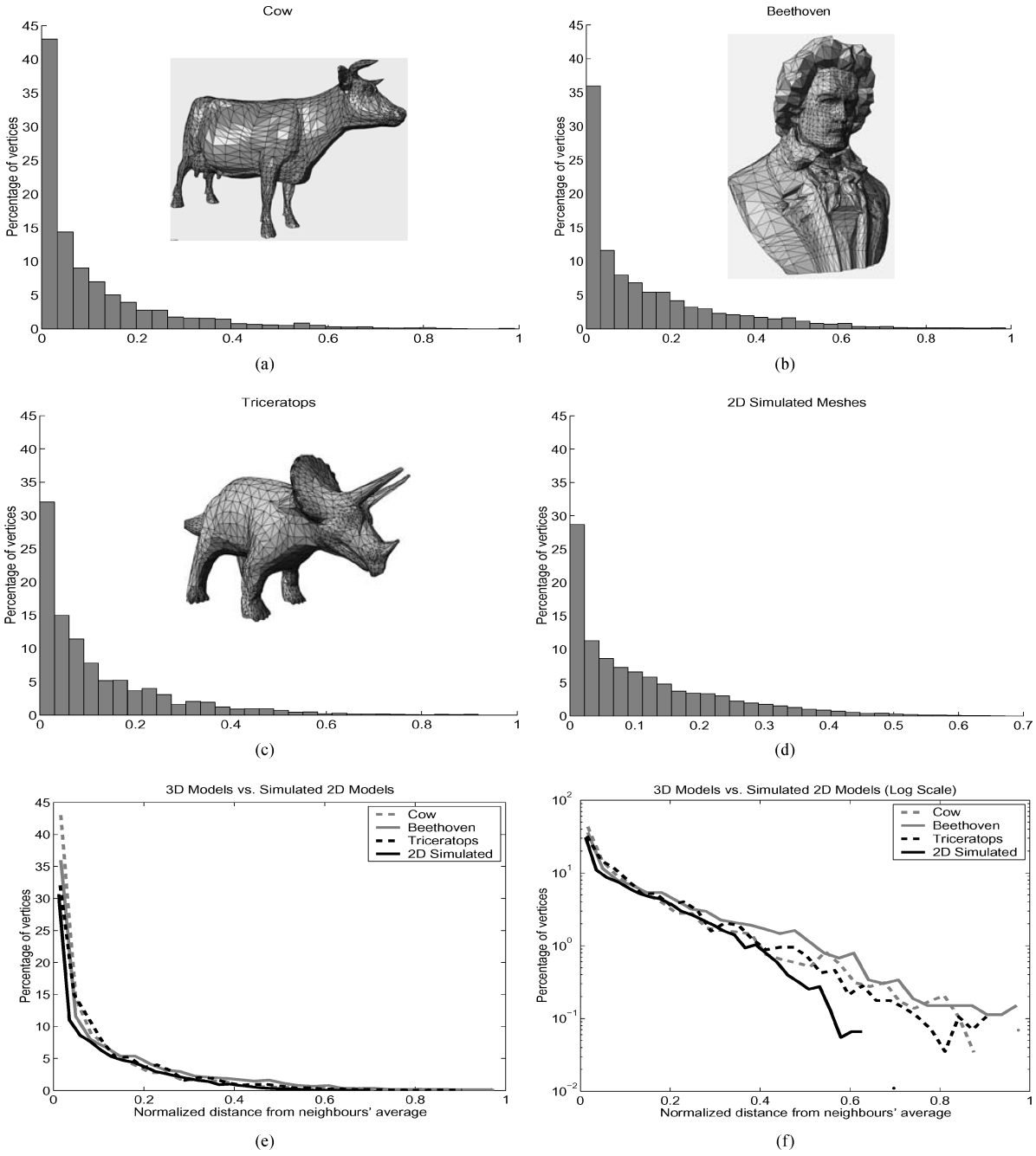


Fig. 8. The "energy" contained in the first k coefficients, for a mesh with $n = 1000$ vertices. The graph shows $\sum_{i=1\ldots k} c_i^2 / \sum_{i=1..n} c_i^2$, where c_i are the coefficients of the decomposition by the PCA basis (computed using the empirical covariance matrix), and the spectral basis. Note that this figure shows the coefficients' energy for a *single* mesh, whereas our theory relates to the similarity of the coefficients on the *average* over *many* possible valid meshes.

On the practical side, to achieve an optimal decomposition, the geometries should be centered to have a zero mean, before the decomposition. This is crucial since, as described in Section 3, PCA is optimal only for a random vector X, which has a zero mean. The geometries as we described them do not have such a property, therefore they should be centered before decomposition by subtracting away their mean, which can be shown to be $Exp(X) = L^{-1}B_X$—the "Tutte configuration" with the given boundary, namely, where each interior vertex is at the centroid of its neighbors.


Cow
Percentage of vertices
(a)
Beethoven
Percentage of vertices
(b)
Triceratops
Percentage of vertices
(c)
2D Simulated Meshes
Percentage of vertices
(d)
3D Models vs. Simulated 2D Models
Cow
Beethoven
Triceratops
2D Simulated
Percentage of vertices
Normalized distance from neighbours' average
(e)
3D Models vs. Simulated 2D Models (Log Scale)
Cow
Beethoven
Triceratops
2D Simulated
Percentage of vertices
Normalized distance from neighbours' average
(f)

There are a few more directions that are worth exploring. These are described in the next sections.

6.1 Other Distributions

Our proof is based heavily on our assumptions about the distribution of meshes in one and two dimensions. The distributions that we impose are obviously not the only ones possible. In one dimension, one may use the same scheme as in two dimensions, and generate the geometries using a random barycentric coordinates matrix. Another possibility is to define X_i to be $\sum_{j=1}^{i} Y_j / \sum_{j=1}^{n} Y_j$, where Y_i are uniformly distributed random variables. This is guaranteed to generate a monotonically increasing sequence. The optimality proof in the one-dimensional case hinges on the key fact that the random variables $X_i - X_{i-1}$ are interchangeable, hence the proof holds for any distribution that satisfies that condition. Specifically, this is the case for both of the models just described, even though these do not generate uniform distributions on the class of valid geometries. It is encouraging to see that the optimality result is not too sensitive to the geometry distribution.

In two dimensions, there are two main variations on our model. One possibility is not to use barycentric coordinates at all: For example, one can generate random X and Y coordinates, and keep only the (X, Y) vector pairs that form a valid geometry. A geometry will be valid if all the triangles have the same orientation. Obviously, this is not an efficient method to sample this distribution, since for large n, the probability that a random (X, Y) pair will form a valid geometry is very small. The advantage of this process, however, is that it will generate geometries distributed uniformly over the class of valid geometries, but it is not clear whether our proof extends to this distribution.

Another possibility is to generate random 2D geometries by modifying the distribution of the barycentric coordinates matrix. For example, instead of being uniform spacings, the barycentric coordinates can be $w_{ij} = Y_i / \Sigma_{i=1..d} Y_i$, where Y_i are independent, identically distributed, uniform random variables. However, here too it is not clear whether the optimality result will still hold.

Although our 2D optimality result was derived using properties of the normal distribution, which is just an approximation of the true geometry distribution, we believe that the result actually holds for the true distribution of the geometries, as in the 1D case, without resorting to normal approximations. This, unfortunately, will probably require a completely different proof.

For both 1D and 2D meshes, our proof is based on the fact that the inverse covariance matrix has the same eigenvectors as the Laplacian, and there is no constraint on the eigenvalues as long as the reverse ordering is preserved. This fact implies another way to apply our proof to other distributions: one can prove that the inverse covariance equals to some integer power of the Laplacian, or any other function of the Laplacian that doesn't change the eigenvectors and the order of the eigenvalues of the resulting matrix.

6.2 The Decay Rate of the Spectral Coefficients

For compression applications, it is important to know how fast the spectral coefficients decay to zero. This property relates directly to the rate/distortion relation of the compression, and is also known as "*Energy Packing Efficiency*". Based on our proof that the Laplacian equals the inverse covariance up to a constant, we can show that the spectral coefficients for the meshes treated here decrease on the average. This, in itself, is an important fact, since *a priori* there is no reason they should even decrease. Let X

←

Fig. 9. The empirical distribution of the relative distance of a vertex from the average of its neighbours: $\frac{(x_i - Avg(x_j))^2}{Avg((x_i - x_j)^2)} | j \in Neigh(i)$, where x_i is the value of the X coordinate of the i-th vertex. The graphs show the histogram of this value, for (a) the "Cow" model, (b) the "Beethoven" model, (c) the "Triceratops" model, and (d) 2D geometries sampled from the distribution described in this article. All four distributions (a-d) are on the same plot, using linear (e) and log (f) scale.

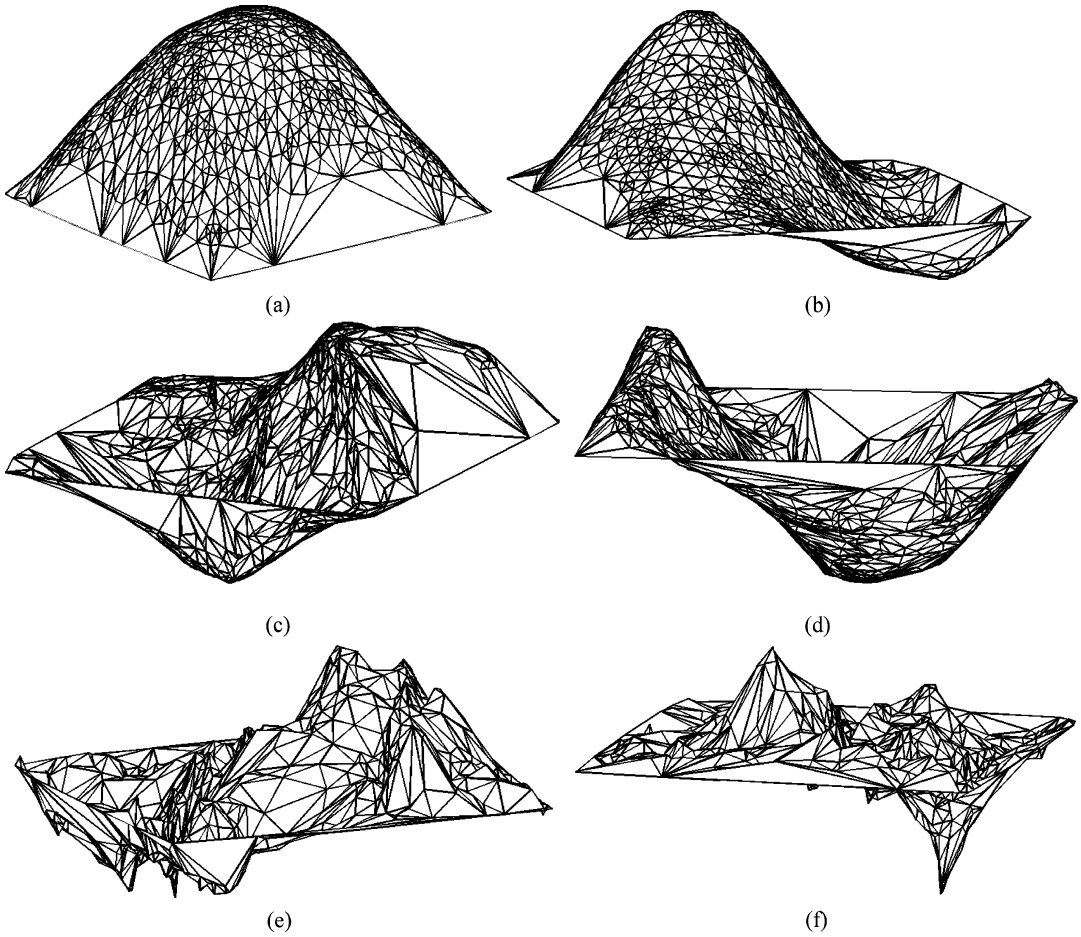


Fig. 10. 3D geometries with common connectivity and boundaries. (a) The barycentric coordinates matrix is $(I - L')$, where L' is the Laplacian L normalized to unit row sums, and the displacement has constant variance for all vertices. (b) The barycentric coordinates matrix is $(I - L')$, and the displacement is smooth. It was generated by randomizing displacement values on the boundaries, and computing the displacement at each vertex by $N = L^{-1}B_N$. (c, d) The displacement is computed as in (b), but random barycentric coordinates matrices are used (e, f) Random barycentric coordinates matrix and random Gaussian displacement with variance $1/n^2$, where n is the number of vertices in the mesh.

be a random geometry (column) vector, and $C = \Psi^T X$ the corresponding spectral coefficient vector. By definition, Ψ is the orthogonal matrix of the column eigenvectors of L—the mesh Laplacian—in increasing order of the eigenvalues: $L\Psi = \Psi\Delta$, where $\Delta = \text{diag}(\mu_1, .., \mu_n)$ is the diagonal matrix of L's eigenvalues, in that order. We have proven that the geometry covariance matrix is $Cov(X) = \alpha L^{-1}$. Hence $Cov(C) = Cov(\Psi^T X) = \Psi^T Cov(X)\Psi = \alpha\Psi^T L^{-1}\Psi$. Now, since $L^{-1} = \Psi\Delta^{-1}\Psi^T$, we obtain $Cov(C) = \alpha\Psi^T\Psi\Delta^{-1}\Psi^T\Psi = \alpha\Delta^{-1}$. This means that the spectral coefficients are pairwise uncorrelated and decrease on the average. Since, by definition, $Exp(X) = 0$, this implies that also $Exp(C) = 0$, so the variance of C is $(\alpha/\mu_1, .., \alpha/\mu_n)$, which obviously decreases.

For the 1D, case, the exact eigenvalues of L are known, so we can find the decay rate of these coefficients. In 1D, the Laplacian is a symmetric tridiagonal Toeplitz matrix, whose eigenvalues are known [Hartfiel and Mayer 1998] to be $\mu_i = 4\sin^2(\frac{\pi i}{2(n+1)})$. For large n, the argument of the sin function

is very small, so the sin function may be approximated by its argument. This means that the inverse eigenvalues, and hence the spectral coefficients, decay like $\theta(1/i^2)$.

6.3 Extension to 3D Meshes

In three dimensions, matters are more complicated. Just applying the barycentric method (15) to a (nonplanar) 3D boundary results in a smooth, featureless surface interpolating the boundary vertices. This is obviously not rich enough to capture interesting 3D shapes.

A possible natural extension of the 2D model, which allows for 3D features to emerge, is the following linear system: $X = WX + N$, where N is a random displacement vector, independent of X, and the system has fixed boundary vertices. This means that each vertex is a convex combination of its neighbors up to a displacement. The displacement can be any reasonable random variable, as long as it is independent of X. The variance of the displacement values will influence the smoothness of the mesh. A smooth mesh can be created by randomizing displacement values on the boundaries, and then computing the displacement values on the internal vertices by computing $N = L^{-1}B_N$, where L is the Laplacian, and B_N are the displacement values on the boundaries. The barycentric coordinates matrix W can be generated by uniform spacings, as in the two-dimensional case. See Figure 10 for examples of 3D meshes with different displacements and different barycentric coordinates matrices based on this model.

The optimality proof for the 2D case is based on the following two key properties of the multivariate random variable X: the normal distribution of the conditionals, and the linear conditional expectation. Both those properties carry over to 3D, when using the model we have proposed, due to the displacements N being independent of X.

Obviously, such a 3D model is not really satisfying, since it cannot generate "real-life" meshes—no cows will be born from random barycentric coordinates and displacements. An interesting experiment may be to generate a family of geometries from a connectivity, for example by using the "Connectivity Shapes" method [Isenburg et al. 2001]. Analyzing the distribution of geometries generated this way may provide an insight for finding the optimal basis for "real-life" 3D meshes. Unfortunately, such an analysis may be mathematically complex.

REFERENCES

ALEXA, M. AND MÜLLER, W. 2000. Representing animations by principal components. *Comput. Graph. For. 19*, 3 (Aug.), 411–418.

ALPERT, C. J. AND YAO, S. Z. 1995. Spectral partitioning: The more eigenvectors, the better. *32th ACM/IEEE Design Automation Conference*, San Francisco, CA. (June) 195–200.

ARNOLD, B. C., CASTILLO, E., AND SARABIA, J. 1999. *Conditional Specification of Statistical Models*. Springer-Verlag.

BESAG, J. AND KOOPERBERG, C. 1995. On conditional and intrinsic autoregressions. *Biometrika, 82*, 4, 733–746.

BERA, A. K. AND JARQUE, C. M. 1981. An efficient large-sample test for normality of observations and regression residuals. Working Paper in Economics and Econometrics 40, Australian National University.

CHUNG, F. R. K. 1997. Spectral graph theory. *Conference Board of Mathematical Science 92*. American Mathematical Society.

CHUNG, F. R. K. AND YAU, S. T. 2000. Discrete Green's functions. *J. Combinator. Theo. A 91* 191–214.

DAVID, H. A. 1981. *Order Statistics*. 2nd Ed. Wiley, New York.

FLOATER, M. S. 1997. Parameterization and smooth approximation of surface triangulations. *Comput. Aided Geometr. Design, 14*, 231–250.

GOTSMAN, C., GU, X., AND SHEFFER, A. 2003. Fundamentals of spherical parameterization for 3D meshes. In *Proceedings of ACM SIGGRAPH 2003*.

GOTSMAN, C. 2003. On graph partitioning, spectral analysis, and digital mesh processing. In *Proceedings of Solid Modeling International*.

HALL, K. M. 1970. An r-dimensional quadratic placement algorithm. *Manage. Science 17*, 219–229.

HARTFIEL, D. J. AND MEYER, C. D. 1998. On the structure of stochastic matrices with a subdominant eigenvalue near 1. *Linear Algebra and Its Appl. 272*, 193–203.

ISENBURG, M., GUMHOLD, S., AND GOTSMAN, C. 2001. Connectivity shapes. In *Proceedings of Visualization 2001.*
JOLLIFFE, I. T. 1986. *Principal Component Analysis*. Springer-Verlag, New-York.
LAURITZEN, S. 1996. *Graphical Models*. Oxford University Press.
KARNI, Z. AND GOTSMAN, C. 2000. Spectral compression of mesh geometry. In *Proceedings of ACM SIGGRAPH 2000*, 279–286.
KOREN, Y. 2003. On spectral graph drawing. In *Proceedings of International Computing and Combinatorics Conference (COCOON'03)*. Lecture Notes in Computer Science, Vol. 2697, Springer Verlag.
PRESS, W. H., TEUKOLSKY, S. A., VETTERLING, W. T., AND FLANNERY, B. P. 1987. *Numerical Recipes*. Cambridge University Press.
PYKE, R. 1965. Spacings. *J. Royal Statist. Soc. B*, 395–439.
RAO, K. R. AND YIP, P. 1990. *Discrete Cosine Transform*. Academic Press.
RICE, J. A. 1988. *Mathematical Statistics and Data Analysis*. Brooks Cole Publishing.
ROSSIGNAC, J. 1999. EdgeBreaker: Connectivity compression for triangle meshes. *IEEE Trans. Visual. Comput. Graph.*, 47–61.
TAUBIN, G. 1995. A signal processing approach to fair surface design. *Proceedings of ACM SIGGRAPH '1995*, 351–358.
TAUBIN, G. AND ROSSIGNAC, J. 1998. Geometric compression through topological surgery. *ACM Trans. Graph. 17*, 2, 84–115.
TOUMA, C. AND GOTSMAN, C. 1998. Triangle mesh compression. In *Proceedings of Graphics Interface*, 26–34.
TUTTE, W. T. 1963. How to draw a graph. In *Proceedings of the London Mathematical Society 13*, 3, 743–768.

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