

PH-CPF: Planar Hexagonal Meshing using Coordinate Power Fields - Supplemental Material

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We present a new approach for computing planar hexagonal meshes that approximate a given surface, represented as a triangle mesh. Our method is based on two novel technical contributions. First, we introduce *Coordinate Power Fields*, which are a pair of tangent vector fields on the surface that fulfill a certain *continuity* constraint. We prove that the fulfillment of this constraint guarantees the existence of a seamless parameterization with quantized rotational jumps, which we then use to regularly remesh the surface. We additionally propose an optimization framework for finding Coordinate Power Fields, which also fulfill additional constraints, such as alignment, sizing and bijectivity. Second, we build upon this framework to address a challenging meshing problem: planar hexagonal meshing. To this end, we suggest a combination of conjugacy, scaling and alignment constraints, which together lead to planarizable hexagons. We demonstrate our approach on a variety of surfaces, automatically generating planar hexagonal meshes on complicated meshes, which were not achievable with existing methods.

Additional Key Words and Phrases: geometry processing, planar hexagonal meshing, parameterization, tangent vector fields

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1 FROM COORDINATE VECTOR FIELDS TO GRADIENT VECTOR FIELDS

Let $r : \mathcal{U} \subset \mathbb{R}^2 \rightarrow \Omega \subset \mathcal{M}$ be a regular parameterization of a planar domain to a patch on the surface. Let \hat{u}, \hat{v} be the Cartesian unit orthogonal axes in the parameterization domain, and u, v be the coordinate functions on \mathcal{U} , and set $U = dr(\hat{u}), V = dr(\hat{v})$, where dr is the differential of r . Let B_p be a local orthonormal basis of $T_p\mathcal{M}$ at $p \in \Omega$, $(U_p \ V_p)$ be the 2×2 matrix whose columns are the coefficients of U_p, V_p in the basis B_p .

LEMMA 1.1. *Let $X \in T_p\mathcal{M}$. Then we have:*

$$dr^{-1}(X) = (U_p \ V_p)^{-1}X, \quad \forall X \in T_p\mathcal{M},$$

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and

$$(\nabla u)_p = (1 \ 0)(U_p \ V_p)^{-1}, \quad (\nabla v)_p = (0 \ 1)(U_p \ V_p)^{-1},$$

where all the coefficients of vectors in $T_p\mathcal{M}$ are with respect to the local basis B_p .

PROOF. Since r is regular and U, V are its coordinate vector fields, then U_p, V_p are linearly independent and $(U_p \ V_p)$ is invertible. Let $(a, b) = (U_p \ V_p)^{-1}X$, or equivalently $X = (U_p \ V_p)(a, b) = aU_p + bV_p$. We additionally have $dr^{-1}(X) = adr^{-1}(U_p) + bdr^{-1}(V_p) = a\hat{u} + b\hat{v} = (a, b)^T$, where the first equality holds since dr^{-1} is linear and the second holds by the definition of U_p, V_p . Thus, we have, as required, $dr^{-1}(X) = (U_p \ V_p)^{-1}X$.

To prove the second equation, consider again $X \in T_p\mathcal{M}$. We have that $\langle \nabla u, X \rangle_p = \langle \nabla u, dr^{-1}(X) \rangle_{r^{-1}(p)} = \langle \hat{u}, dr^{-1}(X) \rangle = (1 \ 0)dr^{-1}(X) = (1 \ 0)(U_p \ V_p)^{-1}X$. Here, the first equality holds since the inner product with the gradient of a function commutes with the pullback of the function. Again, since this holds for any X we have, as required, $\nabla u = (1 \ 0)(U_p \ V_p)^{-1}$. The proof for ∇v is similar. \square

2 CONJUGACY CONDITION

We refer to the notations in Figure 8 (right), where the hexagon is in the tangent plane of the central point p_0 , the coordinates are with respect to U_{p_0}, V_{p_0} , and the conjugacy is with respect to S_{p_0} . We drop the notation of the point to reduce clutter. The three conjugacy conditions corresponding to the three strips of hexes are:

$$(A) \quad 0 = \langle \bar{p}_1 + \bar{p}_6, \bar{p}_1 - \bar{p}_6 \rangle_S = \langle U, \frac{1}{\sqrt{3}}V \rangle_S \Rightarrow \langle U, V \rangle_S = 0,$$

$$(B) \quad 0 = \langle \bar{p}_1 + \bar{p}_2, \bar{p}_1 - \bar{p}_2 \rangle_S = \frac{1}{4} \langle U + \sqrt{3}V, -U + \frac{1}{\sqrt{3}}V \rangle_S = \frac{1}{4} (-\langle U, U \rangle_S - \sqrt{3}\langle V, U \rangle_S + \frac{1}{\sqrt{3}}\langle U, V \rangle_S + \langle V, V \rangle_S). \\ \langle U, V \rangle_S = 0 \Rightarrow \langle U, U \rangle_S = \langle V, V \rangle_S.$$

$$(C) \quad 0 = \langle \bar{p}_2 + \bar{p}_3, \bar{p}_2 - \bar{p}_3 \rangle_S = \frac{1}{4} \langle -U + \sqrt{3}V, U + \frac{1}{\sqrt{3}}V \rangle_S = \frac{1}{4} (-\langle U, U \rangle_S + \sqrt{3}\langle V, U \rangle_S - \frac{1}{\sqrt{3}}\langle U, V \rangle_S + \langle V, V \rangle_S). \\ \langle U, V \rangle_S = 0 \Rightarrow \langle U, U \rangle_S = \langle V, V \rangle_S. \tag{1}$$

Hence, conditions (B), (C) are identical, and given condition (A) they both reduce to $\langle U, U \rangle_S = \langle V, V \rangle_S$, as required.

3 PROOF OF THEOREM 3.5

THEOREM 3.5. *Let U, V be discrete CPFs of degree N . Then there exist functions $u^l, v^l \in \mathbb{R}^m$, $l = [1, 2, 3]$, with u_i, v_i piecewise linear per face $t_i \in \mathcal{F}$ (yet discontinuous between faces), such that:*

- (1) $\nabla(u_i) = (1 \ 0)(U_i \ V_i)^{-1}$, $\nabla(v_i) = (0 \ 1)(U_i \ V_i)^{-1}$, $\forall t_i \in \mathcal{F}$.
- (2) The triangle $t_i = (p_i^1, p_i^2, p_i^3) \in \mathbb{R}^{2 \times 3}$ with coordinates $p_i^l = (u_i^l, v_i^l)$ is positively oriented.
- (3) Let $e_{ij} \in \mathcal{E}^I$, with $e_{ij} = t_i \cap t_j$, and set $\alpha_i, \beta_i \in [1, \dots, 3]$ the indices of the vertices of e_{ij} in t_i , and similarly for t_j . Thus, $e_{ij} = p_i^{\alpha_i} - p_i^{\beta_i} = p_j^{\alpha_j} - p_j^{\beta_j} \in \mathbb{R}^3$. Then there exists $k_{ij} \in \mathbb{Z}$ such that $R^{2\pi k_{ij}/N}(p_i^{\alpha_i} - p_i^{\beta_i}) = p_j^{\alpha_j} - p_j^{\beta_j}$.

PROOF. Let $t_i \in \mathcal{F}$, with embedding $(p_i^1, p_i^2, p_i^3) \in \mathbb{R}^{3 \times 3}$. Given U_i, V_i , set

$$t_i = (p_i^1, p_i^2, p_i^3) = \left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, dr_i^{-1} e_i^{21}, dr_i^{-1} e_i^{31} \right) = \left(\begin{pmatrix} u_i^1 \\ v_i^1 \end{pmatrix}, \begin{pmatrix} u_i^2 \\ v_i^2 \end{pmatrix}, \begin{pmatrix} u_i^3 \\ v_i^3 \end{pmatrix} \right),$$

where $e_i^{21} = p_i^2 - p_i^1$ and similarly for e_i^{31} .

- (1) By the definition of t_i , we have that $((1 \ 0) dr_i^{-1} e_i^{21} = u_i^2 - u_i^1$ and $((1 \ 0) dr_i^{-1} e_i^{31} = u_i^3 - u_i^1$. Since ∇u_i has the same projections on e_i^{21}, e_i^{31} , and since the projections uniquely define the vector as e_i^{21}, e_i^{31} are linearly independent (since the mesh M is positively oriented and not degenerate), we have the result. The claim for v is similar.
- (2) $\det(dr_i^{-1} e_i^{21}, dr_i^{-1} e_i^{31}) = \det(dr_i^{-1}) \det((e_i^{21} \ e_i^{31})) > 0$, and thus t_i is positively oriented. Note that $\det(dr_i^{-1}) > 0$ since U, V are LICO, and $\det((e_i^{21} \ e_i^{31})) > 0$ since the input mesh is positively oriented.
- (3) The two triangles t_i and t_j have coordinates:

$$\begin{aligned} t_i &= (p_i^1, p_i^2, p_i^3) = \left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, dr_i^{-1} e_i^{21}, dr_i^{-1} e_i^{31} \right), \\ t_j &= (p_j^1, p_j^2, p_j^3) = \left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, dr_j^{-1} e_j^{21}, dr_j^{-1} e_j^{31} \right). \end{aligned}$$

Thus, an edge $e_i^{\alpha\beta} = p_i^\alpha - p_i^\beta$ of t_i is given by $e_i^{\alpha\beta} = dr_i^{-1} e_i^{\alpha\beta}$, and similarly for t_j . Let $e_{ij} = p_i^{\alpha_i} - p_i^{\beta_i} = p_j^{\alpha_j} - p_j^{\beta_j}$. Then, from the CPF constraint, we have $(dr_i^{-1}(p_i^{\alpha_i} - p_i^{\beta_i}))^N = (dr_j^{-1}(p_j^{\alpha_j} - p_j^{\beta_j}))^N$, and thus $(p_i^{\alpha_i} - p_i^{\beta_i})^N = (p_j^{\alpha_j} - p_j^{\beta_j})^N$, and the result follows. \square

4 GRADIENT OF THE CPF PENALTY OBJECTIVE

We have:

$$E_{ij}^{\Psi_c}(U, V, z) = |(dr_i^{-1}(e_{ij}))^N - z_{ij}|^2 + |(dr_j^{-1}(e_{ij}))^N - z_{ij}|^2,$$

where, by Equation (4), we have $dr_i^{-1} = \frac{1}{s_i} (-V_i \ U_i)^T J^T$, and $s_i = \langle U_i, -JV_i \rangle$. Here the 2D vectors are considered as complex numbers for the definition of the N -th power and the absolute value. Clearly, the objective $E_{ij}^{\Psi_c}$ is local in the faces i, j , and thus we only compute derivatives with respect to U_i, V_i, U_j, V_j and z_{ij} . Since the expressions are the same for i and j , we provide only the expressions for i .

Define the following auxiliary functions:

$$\begin{aligned} h(w, z) : \mathbb{R}^{2 \times 2} &\rightarrow \mathbb{R}^2, & h(w, z) &= w - z, \\ f(a, b) : \mathbb{R}^2 &\rightarrow \mathbb{R}^2, & f(a, b) &= (\operatorname{Re}((a+ib)^6) \ \operatorname{Im}((a+ib)^6)), \\ s(x, y) : \mathbb{R}^{2 \times 2} &\rightarrow \mathbb{R}, & s(x, y) &= \langle x, -Jy \rangle, \\ g_e(x, y) : \mathbb{R}^{2 \times 2} &\rightarrow \mathbb{R}^2, & g_e(x, y) &= \frac{1}{s(x, y)} (-y \ x)^T J^T e. \end{aligned} \quad (2)$$

Then, the objective for the face i , with respect to the edge e_{ij} between faces i, j is:

$$E_{ij}^{\Psi_c}(U_i, V_i, z_{ij}) = \|h(f(g_{e_{ij}}(U_i, V_i)), z_{ij})\|^2. \quad (3)$$

To reduce clutter, we drop all the subscripts, setting: $x = U_i, y = V_i, z = (\operatorname{Re}(z_{ij}) \ \operatorname{Im}(z_{ij}))$. Then:

$$E(x, y, z) = \|h(f(g(x, y)), z)\|^2. \quad (4)$$

Since we have $\partial E = 2h \partial h$, we detail only the derivatives of h . Set $w(x, y) = f(g(x, y))$:

$$\frac{\partial}{\partial z} h(w(x, y), z) = -\mathbb{I}_2 \quad (5)$$

$$\begin{aligned} \frac{\partial}{\partial x} h(w(x, y), z) &= \frac{\partial h}{\partial w} \frac{\partial w(x, y)}{\partial x} = \mathbb{I}_2 \frac{\partial w(x, y)}{\partial x} = \frac{\partial w(x, y)}{\partial x}, \\ \frac{\partial}{\partial y} h(w(x, y), z) &= \frac{\partial h}{\partial w} \frac{\partial w(x, y)}{\partial y} = \mathbb{I}_2 \frac{\partial w(x, y)}{\partial y} = \frac{\partial w(x, y)}{\partial y}. \end{aligned} \quad (6)$$

Now for the derivatives of $w = f \circ g$, we have:

$$\begin{aligned} \frac{\partial w(x, y)}{\partial x} &= \frac{\partial f(g(x, y))}{\partial x} = J_f(g(x, y)) \frac{\partial g(x, y)}{\partial x}, \\ \frac{\partial w(x, y)}{\partial y} &= \frac{\partial f(g(x, y))}{\partial y} = J_f(g(x, y)) \frac{\partial g(x, y)}{\partial y}, \end{aligned} \quad (7)$$

where $J_f, \partial g/\partial x, \partial g/\partial y \in \mathbb{R}^{2 \times 2}$.

4.1 Jacobian of f

We first write f it explicitly in terms of its real variables. Consider the polar representation $a + ib = r \cos \theta + ir \sin \theta$. Then, $(a + ib)^6 = r^6 \cos 6\theta + ir^6 \sin 6\theta = x + iy$. Now, we have [?, Ch. 6.1.13]:

$$\begin{aligned} \cos 6\theta &= 32 \cos^6 \theta - 48 \cos^4 \theta + 18 \cos^2 \theta - 1 \\ \sin 6\theta &= \sin \theta (32 \cos^5 \theta - 32 \cos^3 \theta + 6 \cos \theta). \end{aligned} \quad (8)$$

Hence:

$$\begin{aligned} x &= r^6 \cos 6\theta = 32r^6 \cos^6 \theta - 48r^2 r^4 \cos^4 \theta + 18r^4 r^2 \cos^2 \theta - r^6, \\ y &= r^6 \sin 6\theta = r \sin \theta (32r^5 \cos^5 \theta - 32r^2 r^3 \cos^3 \theta + 6r^4 r \cos \theta). \end{aligned} \quad (9)$$

Plugging in $a = r \cos \theta, b = r \sin \theta$ we get:

$$\begin{aligned} x &= 32a^6 - 48r^2 a^4 + 18r^4 a^2 - r^6, \\ y &= b (32a^5 - 32r^2 a^3 + 6r^4 a). \end{aligned} \quad (10)$$

Finally, plugging in $r^2 = a^2 + b^2$ we have:

$$\begin{aligned} x &= 32a^6 - 48(a^2 + b^2)a^4 + 18(a^2 + b^2)^2 a^2 - (a^2 + b^2)^3, \\ y &= b (32a^5 - 32(a^2 + b^2)a^3 + 6(a^2 + b^2)^2 a). \end{aligned} \quad (11)$$

After simplification we get:

$$\begin{aligned} x &= a^6 - 15a^4b^2 + 15a^2b^4 - b^6, \\ y &= 2ab(3a^4 - 10a^2b^2 + 3b^4). \end{aligned} \quad (12)$$

Equation (??) provides an expression in real numbers of the output of f in terms of its input. The derivatives follow:

$$\begin{aligned} \frac{\partial x}{\partial a} &= \frac{\partial y}{\partial b} = 6a(a^4 - 10a^2b^2 + 5b^4), \\ \frac{\partial x}{\partial b} &= -\frac{\partial y}{\partial a} = -6b(5a^4 - 10a^2b^2 + b^4). \end{aligned} \quad (13)$$

Note that if complex numbers are used for the implementation, and $f(z) = z^6$, then the complex derivative is $\frac{\partial f}{\partial z} = 6z^5$, and the real derivative is then $\frac{\partial x}{\partial a} = \frac{\partial y}{\partial b} = \text{Re}(\frac{\partial f}{\partial z})$ and $\frac{\partial x}{\partial b} = -\frac{\partial y}{\partial a} = -\text{Im}(\frac{\partial f}{\partial z})$.

4.2 Jacobian of g

By the definition of g we have that $g(x, y) = (x \ y)^{-1}e$, where e is constant with respect to x, y . Given a matrix A we have [?, Ch. 5.1.10.2.5]

$$\frac{\partial(A^{-1}B)}{\partial A_{ij}} = -A^{-1}E_{ij}A^{-1}B, \quad (14)$$

where E_{ij} is a matrix which is zero everywhere except at (i, j) , where it is 1.

Let $A^{-1} = \begin{pmatrix} p \\ q \end{pmatrix}$, where $p, q \in \mathbb{R}^{1 \times 2}$. Then:

$$\begin{aligned} \frac{\partial(A^{-1}B)}{\partial A_{11}} &= -A^{-1} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} A^{-1}B = -A^{-1} \begin{pmatrix} pB \\ 0 \end{pmatrix}, \\ \frac{\partial(A^{-1}B)}{\partial A_{21}} &= -A^{-1} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} A^{-1}B = -A^{-1} \begin{pmatrix} 0 \\ pB \end{pmatrix}, \\ \frac{\partial(A^{-1}B)}{\partial A_{12}} &= -A^{-1} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} A^{-1}B = -A^{-1} \begin{pmatrix} qB \\ 0 \end{pmatrix}, \\ \frac{\partial(A^{-1}B)}{\partial A_{22}} &= -A^{-1} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} A^{-1}B = -A^{-1} \begin{pmatrix} 0 \\ qB \end{pmatrix}. \end{aligned} \quad (15)$$

Hence:

$$\frac{\partial(A^{-1}B)}{\partial A_{:,1}} = -(pB)A^{-1}, \quad \frac{\partial(A^{-1}B)}{\partial A_{:,2}} = -(qB)A^{-1}. \quad (16)$$

We have that $(xy)^{-1} = \frac{1}{s(x, y)} \begin{pmatrix} -Jy^T \\ Jx^T \end{pmatrix}$, therefore:

$$\begin{aligned} \frac{\partial g(x, y)}{\partial x} &= \frac{e^T Jy}{s^2(x, y)} (-y \ x)^T J^T = -((1 \ 0)g(x, y))(x \ y)^{-1}, \\ \frac{\partial g(x, y)}{\partial y} &= -\frac{e^T Jx}{s^2(x, y)} (-y \ x)^T J^T = -((0 \ 1)g(x, y))(x \ y)^{-1}. \end{aligned} \quad (17)$$

5 GRADIENT OF THE SMOOTHNESS OBJECTIVE

Similarly to the continuity constraint, we have:

$$E_{ij}^s(U, V) = |(dr_i^{-1}(Je_{ij}))^N - (dr_j^{-1}(Je_{ij}))^N|^2,$$

where, by Equation (4), we have $dr_i^{-1} = \frac{1}{s_i} (-v_i \ U_i)^T J_i^T$, and $s_i = \langle U_i, -JV_i \rangle$. Clearly, the objective E_{ij}^s is local in the faces i, j , and thus we only compute derivatives with respect to U_i, V_i, U_j, V_j . Using the

same auxiliary functions h, f, s as in Equation (??), and redefining g as:

$$g_e(x, y) : \mathbb{R}^{2 \times 2} \rightarrow \mathbb{R}^2, \quad g_e(x, y) = \frac{1}{s(x, y)} (-y \ x)^T e, \quad (18)$$

we have

$$E_{ij}^s(U_i, V_i, U_j, V_j) = \left\| h \left(f(g_{e_{ij}}(U_i, V_i)), f(g_{e_{ij}}(U_j, V_j)) \right) \right\|^2. \quad (19)$$

Taking again $w(x, y) = f(g(x, y))$, and $x_i = U_i, y_i = V_i, x_j = U_j, y_j = V_j$ we have:

$$E(x_i, y_i, x_j, y_j) = \left\| h \left(w(x_i, y_i), w(x_j, y_j) \right) \right\|^2, \quad (20)$$

and

$$\begin{aligned} \frac{\partial}{\partial x_i} h_{ij} &= w_x(x_i, y_i) = J_f(g(x_i, y_i))g_x(x_i, y_i), \\ \frac{\partial}{\partial y_i} h_{ij} &= w_y(x_i, y_i) = J_f(g(x_i, y_i))g_y(x_i, y_i), \\ \frac{\partial}{\partial x_j} h_{ij} &= -w_x(x_j, y_j) = -J_f(g(x_j, y_j))g_x(x_j, y_j), \\ \frac{\partial}{\partial y_j} h_{ij} &= -w_y(x_j, y_j) = -J_f(g(x_j, y_j))g_y(x_j, y_j), \end{aligned} \quad (21)$$

where $h_{ij} = h(w(x_i, y_i), w(x_j, y_j))$, $w_x = \partial w / \partial x$, $w_y = \partial w / \partial y$, $g_x = \partial g / \partial x$, $g_y = \partial g / \partial y$.

Following Equation (??), the definition of g and since $J^T J = \mathbb{I}_2$, we have:

$$\begin{aligned} g_x &= \frac{e^T y}{s^2(x, y)} (-y \ x)^T J^T = -((1 \ 0)g(x, y))(x \ y)^{-1}, \\ g_y &= -\frac{e^T x}{s^2(x, y)} (-y \ x)^T J^T = -((0 \ 1)g(x, y))(x \ y)^{-1}. \end{aligned} \quad (22)$$

6 PLANARIZATION OPTIMIZATION PROBLEM

We solve the following optimization problem:

$$\begin{aligned} &\underset{v \in \mathbb{R}^{|\mathcal{V}_H| \times 3}, n \in \mathbb{R}^{|\mathcal{F}_H| \times 3}}{\text{minimize}} && \lambda_d E_d + \lambda_s E_s + \lambda_c E_c + \lambda_l E_l \\ &\text{subject to} && \lambda_p \sum_{f \in \mathcal{F}_H} \sum_{e \in f} \langle n_f, e \rangle^2 = 0, \quad (\text{planarity}), \end{aligned} \quad (23)$$

where

$$E_d = \sum_{v_i \in \mathcal{V}_H} \|v_i - \hat{v}_i\|^2 \quad (\text{distance to surface})$$

$$E_s = \sum_{\substack{v_i \in \mathcal{F}_f \\ f \in \mathcal{F}_H}} \|(c_f - v_i) - (v_{i+\frac{|f|}{2}} - c_f)\|^2 \quad (\text{symmetry})$$

$$E_c = \sum_{v_i \in \mathcal{V}_H} \langle n_{\tilde{v}}, (v_i - \tilde{v}_i) \rangle^2 \quad (\text{tangential drift})$$

$$E_l = \sum_{e_{ij} \in \mathcal{E}_H} \phi(\|e_{ij}\|) + \sum_{\substack{v_i \in \mathcal{F}_f \\ f \in \mathcal{F}_H}} \phi(\|v_i - \tilde{v}_{i+\frac{|f|}{2}}\|) \quad (\text{edge and diagonal lengths})$$

with an auxiliary constraint $\forall f \in \mathcal{F}_H, \|n_f\| = 1$, that is enforced using a homogeneous parametrization. Here, c_f is the barycenter of f , \tilde{v} denotes the position of v in the previous iteration and $n_{\tilde{v}}$ is computed by averaging normal vectors at faces adjacent to \tilde{v} . Further, \hat{v} is a projection of \tilde{v} onto the input triangle mesh M , and

e is an edge vector. Finally $\phi(\cdot)$ is a barrier function as defined in Equation (14). For quad meshes, we add the fairness term $\lambda_f E_f = \sum_{v \in \mathcal{V}_Q} \|v - (w_1 + \dots + w_n)/n\|^2$, where w_1, \dots, w_n are the nearest neighbors of v , and $\lambda_f = 5$.

We use the Ceres Solver [?] for the optimization, where we enforce the barrier on the lengths by multiplying λ_l by 2 after every 5 solver iterations. We initialize the weights as follows: $\lambda_p = \lambda_c = 0.01$, $\lambda_l = 0.001$, $\lambda_d = 0.5$, $\lambda_s = 1$. We stop the optimization when

$\max_{f \in \mathcal{F}_H} (\varepsilon_p(f))$ is smaller than a user-prescribed threshold, or when the maximal number of iterations is reached. We let the solver perform a maximum number of 250 internal iterations and then update \tilde{v} , c_f , and $n_{\tilde{v}}$, and the weight λ_p .

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